6.1. Differentiation rules in more variables

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = \int_{\sin(x)}^{\cos(y)} e^{zt} dt.$$

Compute $\nabla f(\frac{\pi}{3}, \frac{\pi}{2}, 0)$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$ exist, and that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

6.2. The geometry of the gradient Let $c \in \mathbb{R}$ be a constant and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant differentiable function. Assume that the equation $f(x, y) = c$ defines a curve C in the plane $\mathbb{R}^2$. I.e. there exists an interval $I \subset \mathbb{R}$ and an injective, differentiable map $\gamma : I \to \mathbb{R}^2$, so that

$$\gamma(I) = C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\} \tag{1}$$

and $\gamma'(t) \neq 0$ fr all $t \in I$. Prove the following statements:

(a) ∇f is perpendicular to C. I.e. for all $t \in I$ we have $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$.

(b) The directional derivative of f in a direction along C vanishes. I.e. $D_{\gamma'(t)} f(\gamma(t)) = 0$ for all $t \in I$.

(c) The directional derivative of f is largest in a direction perpendicular to C.

6.3. Tangential planes Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \sin(x) - y^3 + y^2.$$

(a) Determine the equation of the tangential plane of the surface

$$\mathcal{G}(f) := \{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

at the point $(0, 3, f(0, 3)) = (0, 3, -18)$.

(b) Determine a constant $c \in \mathbb{R}$, so that the vector

$$\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$$

is perpendicular to the surface $\mathcal{G}(f)$ at the point $(\frac{\pi}{2}, 0, 1) \in \mathcal{G}(f)$.