11.1. Fubini's theorem for explicit functions I

(1) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4y - y^5x + y^3\right) dxdy$$

(2) Let $D^2 = \mathbb{R}^2 \cap \{(x,y) : x^2+y^2 \leq 1\}$ be the unit disk in the plan. Compute

$$\int_{D^2} x^2y^2 dxdy$$

by following the following steps.

(a) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \frac{\pi}{32}$$

(b) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x,y)dxdy = \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dy\right)dx.$$

(c) Compute

$$\int_{D^2} x^2y^2dxdy,$$

by making the formula of question (2) and a (1-dimensional) change of variable using trigonometric functions and symmetry.

11.2. Fubini's theorem for explicit functions II

Compute the following double integrals $\int_D f(x,y)dxdy$, where the continuous function $f : D \to \mathbb{R}$ and the domain D are given by

1. $f(x,y) = x$, and $D = \mathbb{R}^2 \cap \{(x,y) : y \geq 0, x-y+1 \geq 0, x+2y-4 \leq 0\}$.
2. $f(x,y) = \cos(xy)$, and $D = \mathbb{R}^2 \cap \{(x,y) : 1 \leq x \leq 2, 0 \leq xy \leq \frac{\pi}{2}\}$.
3. $f(x,y) = \frac{1}{(x+y)^3}$, and $D = \mathbb{R}^2 \cap \{(x,y) : 1 \leq x \leq 3, y \geq 2, x+y \leq 5\}$.
4. $f(x,y) = \frac{xy}{1+x^2+y^2}$, and $D = \mathbb{R}^2 \cap \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2+y^2 \geq 1\}$.

11.3. Fubini's theorem for explicit functions III

Compute the area of the domain

$$D = \mathbb{R}^2 \cap \{(x,y) : -1 \leq x \leq 1, x^2 \leq y \leq 4-x^3\}.$$

11.4. Are the following sets negligible in $\mathbb{R}^3$?

1. $\{(i,j,k) \in \mathbb{R}^3 \mid i,j,k \in \mathbb{Z}, i^2+j^2+k^2 < 2019\}$.
2. $\{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1, x,y \in [0,1]\}$.