12.1. Volume of the 3-dimensional ball Let $r > 0$ and $B_3(0,r) = \mathbb{R}^3 \cap \{(x,y,z) : x^2 + y^2 + z^2 \leq r^2\}$ be the open ball of radius $r > 0$. By using a change of coordinates $f : [0,r) \times [0,2\pi) \times [0,\pi) \to B_3(0,r)$ is given as follows :

$$f(t,\theta,\varphi) = \begin{cases} t\cos(\theta)\sin(\varphi) \\ t\sin(\theta)\sin(\varphi) \\ t\cos(\varphi) \end{cases}$$

Compute the volume of $B_3(0,r)$, defined by

$$\int_{B_3(0,r)} dxdydz.$$

12.2. n-dimensional volume of a ball Let $n \geq 2$, $r > 0$ and

$$B_n(0,r) = \mathbb{R}^n \cap \{x = (x_1, \cdots, x_n) : x_1^2 + \cdots + x_n^2 \leq r^2\}$$

be the closed ball of radius r. Let

$$V(n,r) = \int_{B_n(0,r)} dx_1 \cdots dx_n$$

be the volume of the ball of radius 1 and write for simplicity $V(n) = V(n,1)$ for the volume of the unit ball.

1. Compute $V(1)$ and $V(2)$.
2. Show by a change of variable that for all $r > 0$, we have
$$V(n,r) = r^n V(n).$$
3. Show by Fubini's theorem that for all $n \geq 2$, we have
$$V(n) = \frac{2\pi}{n} V(n-2)$$
4. Deduce that for all $n \in \mathbb{N}$, we have
$$V(2n) = \frac{\pi^n}{n!} \quad V(2n+1) = \frac{2^{2n+1} n!}{(2n+1)!} \pi^n.$$
where we recall that $n! = 1 \times 2 \times \cdots (n-1) \times n$ for all $n \geq 1$.

Hint: Notice in 3. that

$$B_n(0,1) = \mathbb{R}^n \cap \{x = (x_1, \cdots, x_n) : x_{n-1}^2 + x_n^2 \leq 1,\ x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_{n-1}^2 - x_n^2\}$$

12.3. Fubini's theorem for an explicit integral By computing the integral (justify why it converges)

$$I = \int_{[0,\infty) \times [a,b]} e^{-xy} dxdy$$

in two different ways, where $0 < a < b$, compute

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$