1.1. Let $f : \mathbb{R} \to \mathbb{R}^d, d \in \mathbb{N}^*$ be a function defined by

$$f(x) = (f_1(x), f_2(x), ..., f_d(x))$$

where $f_1, f_2, ..., f_d$ are function from $\mathbb{R}$ to $\mathbb{R}$.

(a) Show that if f_1 is injective, then f is injective.

(b) Show that if $d \geq 2$, and f_i is surjective for all i between 1 and d, then f is not always surjective.

Solution:

(a) For any $x, y \in \mathbb{R}$ with $x \neq y$, $f_1(x) \neq f_1(y)$ because f_1 is injective. Thus we also have $f(x) \neq f(y)$, which gives that f is injective.

(b) It suffices to give a counterexample for every $d \geq 2$. We define

$$f_1(x) = f_2(x) = ... = f_d(x) = x,$$

It is obvious that f is not surjective for any $d \geq 2$.

1.2. For each function $f = (f_1, f_2) : [0, 1] \to \mathbb{R}^2$ below, sketch the set

$$\{(f_1(t), f_2(t)) | t \in [0, 1]\}$$

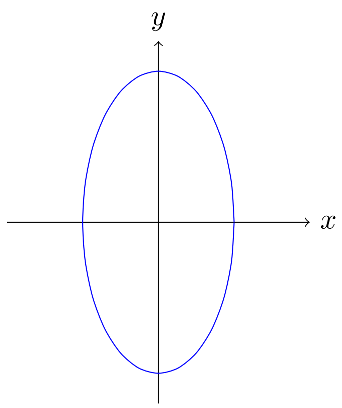
(a) $f(t) = (\cos(2\pi t), 2\sin(2\pi t))$,

(b) $f(t) = (t^2, t^3)$,

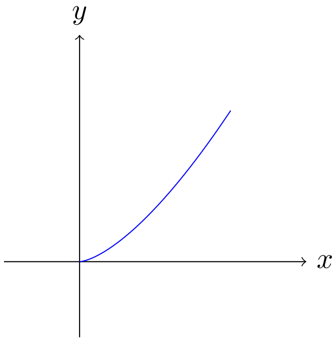
(c) $f(t) = (\frac{t}{1+t^2}, \frac{1}{1+t^2})$.

Solution:

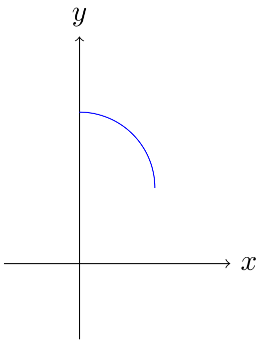
(a) Let $x = \cos(2\pi t)$, $y = 2\sin(2\pi t))$, then we have $4x^2 + y^2 = 4$. Hence we have a complete ellipse.



(b) Let $x = t^2$, $y = t^3$, then $x^3 = y^2$ and $x \in [0, 1]$.



(b) Let $x = \frac{t}{1+t^2}$, $y = \frac{1}{1+t^2}$, then $x^2 + y^2 = y$ and $x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1]$.



1.3. Which of the following are ODEs? Which are linear? Which are linear homogeneous?

(a) $f'(x) = f(x+1)$,

(b) $y^2 = y'y''$,

(c) $y'' + \cos(y)y' + y = x^2$,

(d) $y'' + 2y = -e^{x^2}$,

(e) $y^{(3)} + 6y' + y = 0$.

Solution:

(a) Not an ODE, because it relates values of f and f' at different points (at x and $x+1$).

(b) An ODE but not linear, because of the presence of the quadratic term y^2.

(c) An ODE but not linear, because of the presence of the term $\cos(y)$.

(d) Linear ODE but not homogeneous, because of the presence of $-e^{x^2}$.

(e) Linear homogeneous ODE.

1.4. For each function below, find a simple linear homogeneous ODE for which they are solutions:

(a) $f(x) = e^{x^3}$,

(b) $f(x) = \frac{1}{1+x^2}$,

(c) $f(x) = e^x \cos(x)$.

Solution:

(a) We have $f'(x) = 3x^2 e^{x^3}$, then clearly the ODE can be

$$f'(x) = 3x^2 f(x)$$

(b) We have $f'(x) = -\frac{2x}{(1+x^2)^2}$, then clearly the ODE can be

$$f'(x) = -\frac{2x}{1+x^2} f(x)$$

(c) We have $f'(x) = e^x \cos(x) - e^x \sin(x)$ and $f''(x) = -2e^x \sin(x)$, then clearly the ODE can be

$$f''(x) = 2f'(x) - 2f(x)$$