2.1. Compute the general solutions of following differential equations.

(a) $y''(x) - y'(x) - 2y(x) = 0$,

(b) $y^{(4)}(x) + y(x) = 0$.

Solution:

(a) The characteristic polynomial is

$$X^2 - X - 2 = (X-2)(X+1),$$

then there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$y(x) = \lambda_1 e^{2x} + \lambda_2 e^{-x}.$$

(b) The characteristic polynomial is

$$X^4 + 1 = (X^2 + i)(X^2 - i) = \prod_{k=1}^{4}(X - c_k),$$

where $c_1, c_2 = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$ and $c_3, c_4 = \pm\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$, then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\begin{aligned} y(t) = {} & \mathrm{e}^{\frac{\sqrt{2}}{2}t}\left(\lambda_1 \cos\left(\frac{\sqrt{2}}{2}t\right) + \lambda_2 \sin\left(\frac{\sqrt{2}}{2}t\right)\right) \\ & + \mathrm{e}^{-\frac{\sqrt{2}}{2}t}\left(\lambda_3 \cos\left(\frac{\sqrt{2}}{2}t\right) + \lambda_4 \sin\left(\frac{\sqrt{2}}{2}t\right)\right) \end{aligned}$$

2.2. Use variation of coefficients to compute solutions of the following differential equation.

$$y'' + 4y = \frac{1}{\sin(2x)}$$

Solution:

(a) We look at the characteristic polynomial of the homogeneous problem

$$z^2 + 4 = (z + 2i)(z - 2i).$$

Its general solution is given by

$$y(x) = \lambda_1 \cos(2x) + \lambda_2 \sin(2x)$$

with unknown constants $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume λ_1, λ_2 are two functions and

$$y_0(x) = \lambda_1(x)\cos(2x) + \lambda_2(x)\sin(2x)$$

then

$$\begin{cases} \lambda_1'(x)\cos(2x) + \lambda_2'(x)\sin(2x) = 0, \\ -2\lambda_1'(x)\sin(2x) + 2\lambda_2'(x)\cos(2x) = \dfrac{1}{\sin(2x)} \end{cases} \tag{1}$$

For all $x \in \mathbb{R}$, the matrix

$$R(x) = \begin{pmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{pmatrix}$$

is invertible ($\det R(x) = 2\cos^2(2x) + 2\sin^2(2x) = 2 \neq 0$) and

$$R(x)^{-1} = \frac{1}{2}\begin{pmatrix} 2\cos(2x) & -\sin(2x) \\ 2\sin(2x) & \cos(2x) \end{pmatrix} = \begin{pmatrix} \cos(2x) & -\frac{1}{2}\sin(2x) \\ \sin(2x) & \frac{1}{2}\cos(2x) \end{pmatrix}. \tag{2}$$

Then (1) and (2) give

$$\begin{cases} \lambda_1'(x) = -\dfrac{1}{2} \\ \lambda_2'(x) = \dfrac{\cos(2x)}{2\sin(2x)}. \end{cases}$$

and

$$\begin{cases} \lambda_1(x) = -\dfrac{x}{2}, \\ \lambda_2(x) = \dfrac{1}{4}\log|\sin(2x)|. \end{cases}$$

We can deduce

$$y(x) = \lambda_1\cos(2x) + \lambda_2\sin(2x) - \frac{x}{2}\cos(2x) + \frac{1}{4}\log|\sin(2x)|\sin(2x), \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

2.3. Solve the following differential equations

(a) $y' = (x+y)^2$,

(b) $y' - y = \sin x$,

(c) $y\,y' - (1+y)x^2 = 0$.

Hints: For (1) look at the substitution $z = x + y$. For (2) try to multiply both sides with e^{-x}.

Solution:

(a) we do the substitution $z = x+y$ which gives $z' = 1+y'$. The equation becomes $z' = z^2+1$. We do separation of variable

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z^2 + 1 \Rightarrow \int \frac{\mathrm{d}z}{z^2+1} = \int \mathrm{d}x \Rightarrow \arctan z = x + c, \quad C \in \mathbb{R}.$$

Thus the solutions are given by $z(x) = \tan(x + C)$, $C \in \mathbb{R}$, which leads to $y(x) = \tan(x + C) - x$, $C \in \mathbb{R}$.

(b) we multiply the both sides with $\rho(x) = \mathrm{e}^{-x}$, then this yields

$$\mathrm{e}^{-x}y'(x) - \mathrm{e}^{-x}y = \mathrm{e}^{-x}\sin x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(y(x)\mathrm{e}^{-x}\right) = \mathrm{e}^{-x}\sin x$$
$$\Rightarrow y(x) = \mathrm{e}^{x}\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x + C\mathrm{e}^{x} \quad C \in \mathbb{R}.$$

The indefinite integral can be computed by partial integration:

$$\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x}(\sin x + \cos x),$$

hence the solutions are given by $y(x) = -\frac{1}{2}(\sin x + \cos x) + C\mathrm{e}^{x}$, $C \in \mathbb{R}$.

(c) We observe that $y \equiv -1$ is the only constant solution. Now we look at nonconstant solution. We do separation of variable as following

$$y\,\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y)x^2 \Rightarrow \int \frac{y}{1+y}\,\mathrm{d}y \qquad = \int x^2\,\mathrm{d}x \Rightarrow \int \left(1 - \frac{1}{1+y}\right)\mathrm{d}y = \int x^2\,\mathrm{d}x,$$

Thus the nonconstant solution is gvien by

$$y - \log|1+y| = \frac{x^3}{3} + C, \quad C \in \mathbb{R}.$$

2.4. Let $\omega \in \mathbb{R}^+$. Solve

$$\ddot{x}(t) + \omega^2 x(t) = 0. \tag{3}$$

in the following two cases

(a) $x(0) = 1, \dot{x}(0) = 2\omega$,

(b) $x(0) = 1, x(\frac{\pi}{2\omega}) = 1$.

Solution:

We look at the characteristic polynomial

$$p(\lambda) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega,$$

so we can write down the general solution

$$x_{all}(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \qquad C_1, C_2 \in \mathbb{R}.$$

(a) With the initial condition $x(0) = 1$ and $\dot{x}(0) = 2\omega$ we have

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1\cos(0) + C_2\sin(0) = C_1 = 1, \\ \dot{x}(0) = 2\omega \quad &\Rightarrow \quad -C_1\omega\sin(0) + C_2\omega\cos(0) = C_2\omega = 2\omega. \end{aligned}$$

Thus $C_1 = 1$ und $C_2 = 2$. The solution is then given by

$$x(t) = \cos(\omega t) + 2\sin(\omega t).$$

(b) With the initial condition $x(0) = 1$ and $x(\frac{\pi}{2\omega}) = 1$ we have

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1\cos(0) + C_2\sin(0) = C_1 = 1, \\ x(\frac{\pi}{2\omega}) = 1 \quad &\Rightarrow \quad C_1\cos(\frac{\pi}{2}) + C_2\sin(\frac{\pi}{2}) = C_2 = 1. \end{aligned}$$

Thus $C_1 = C_2 = 1$. The solution is then given by

$$x(t) = \cos(\omega t) + \sin(\omega t).$$