3.1. Which of the following sets are compact?

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 2019\}$;

(b) $B = \{(a, b, c) \in \mathbb{R}^3 \mid a, b, c$ are integers and $a^2 + b^2 + c^2 < 2019\}$;

(c) $C = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in (0, 1], f(x) = \sin(\frac{1}{x})\}$;

(d) $D = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid \theta$ is a rational number$\}$;

(e) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 2\}$.

Solution:

(a) A is bounded but not closed, because the boundary in not included.

(b) B is compact because it contains finitely many points.

(c) C is not closed. $(\frac{1}{2k\pi}, 0) \in C$ for any positive integer k and $(\frac{1}{2k\pi}, 0) \to (0, 0)$ for $k \to +\infty$, but $(0, 0) \notin C$.

(d) D is not closed. Clearly $(0, 1) \notin D$. Indeed, we know $(0, 1) = (\cos\theta, \sin\theta)$ if and only if $\theta = (2k + \frac{1}{2})\pi, k \in \mathbb{Z}$, but $(2k + \frac{1}{2})\pi$ is never a rational number for any $k \in \mathbb{Z}$. Since any irrational can be the limit of a rational sequence, i.e. there exists a rational sequence $\theta_k, k \in \mathbb{N}$ with $\lim_{k\to+\infty} \theta_k = \frac{\pi}{2}$, by continuity we have $\lim_{k\to+\infty}(\cos\theta_k, \sin\theta_k) = (1, 0)$. Hence D is not closed.

(e) E is closed and bounded, hence compact.

3.2. Which of the following functions are continuous?

(a) $f_1(x, y) = \ln(|xy| + 1), (x, y) \in \mathbb{R}^2$;

(b) $f_2(x, y) = \ln(xy), (x, y) \in \{(x, y) \in \mathbb{R}^2 \mid xy > 0\}$;

(c) $f_3(x, y) = \inf\{x^k + y^k \mid k$ is a positive integer$\}, (x, y) \in \mathbb{R}^2$;

(d) $f_4(x, y) = \displaystyle\int_x^y \sin(t)dt, (x, y) \in \{(x, y) \in \mathbb{R}^2 \mid x < y\}$.

Solution:

(a) $(x, y) \to xy$ is clearly continuous in $\mathbb{R}^2$. $x \to \ln(|x| + 1)$ is also continuous. By Proposition 3.2.9, f_1 is continuous.

(b) $(x, y) \to xy$ is clearly continuous in $\{(x, y) \in \mathbb{R}^2 \mid xy > 0\}$ and $x \to \ln x$ is continuous in $\mathbb{R}^+$. By Proposition 3.2.9, f_2 is continuous.

(c) Clearly $f_3(1, 0) = 1$. However, for any $\epsilon \in (0, 1)$, $f_3(1 - \epsilon, 0) = 0$, then f_3 is not continuous.

(d) For any $\epsilon > 0$, if $\|(x_0, y_0) - (x_1, y_1)\| < \epsilon$, then $|x_0 - x_1| < \epsilon$ and $|y_0 - y_1| < \epsilon$. Since $|\sin(t)| < 1$, we know $|f_4(x_0, y_0) - f_4(x_1, y_1)| \leq 2\epsilon$. Hence, f_4 is also continuous.