4.1. Limit. Let f be a function on $\mathbb{R}^2 \backslash \{(0,0)\}$ given by $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$. Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If it exists, compute the limit.

Solution: The $\mathbb{R} \ni t \mapsto \sin(t)$ is continuous and differentiable and

$$\frac{d}{dt}\Big|_{t=0} \sin(t) = \cos(0) = 1.$$

From the definition of derivative, we have

$$\sin(s) = \sin(s) - \sin(0) = \frac{d}{dt}\Big|_{t=0} \sin(t) \cdot s + o(s) = s + o(s), \tag{1}$$

where $s \mapsto o(s)$ is a function such that

$$\frac{o(s)}{s} \xrightarrow{s\to 0} 0 \tag{2}$$

In polar coordinate system we can write $x = r\cos(\theta)$ and $y = r\sin(\theta)$ given $(x,y) \neq (0,0)$, then $f(x,y) = f(r) = \frac{\sin(r^2)}{r^2}$. Hence $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r)$ if the limits exist. With (1) and (2) we have

$$f(r) = \frac{\sin(r^2)}{r^2} = \frac{r^2 + o(r^2)}{r^2} = 1 + \frac{o(r^2)}{r^2},$$

and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\left(1 + \frac{o(r^2)}{r^2}\right) = 1.$$

4.2. Laplace operator. The Laplace operator Δ on $\mathbb{R}^n$ is a differential operator such that for any twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\Delta f = \sum_{i=1}^{n} \partial_{x_i}^2 f \qquad \text{with} \quad \partial_{x_i}^2 f = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_i}\Big).$$

Suppose U is a open set $U \subset \mathbb{R}^n$ and $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is twice differentiable, i.e.

$$\Delta f = \sum_{i=1}^{n} \partial_{x_i}^2 f = 0$$

then we say f is harmonic (on U).

(a) Define $f : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ to be $f(x) = \log|x|$ for any $x \in \mathbb{R}^2 \backslash \{0\}$. Prove f is harmonic.

(b) For $\alpha \in \mathbb{R}$ define $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ to be $f_\alpha(x) = |x|^\alpha$ for any $x \in \mathbb{R}^n \backslash \{0\}$. For which $\alpha \in \mathbb{R}$ is the function f_α harmonic?

The heat operator $\partial_t - \Delta_x$ is defined such that for any twice differentiable function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ we have

$$(\partial_t - \Delta_x)\, f = \partial_t f - \sum_{i=1}^{n} \partial_{x_i}^2 f, \quad (x,t) = ((x_1, \cdots, x_n), t) \in \mathbb{R}^n \times \mathbb{R}.$$

(c) Prove the function $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ defined by

$$u(x,t) = \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \quad \text{for any } (x,t) \in \mathbb{R}^n \times (0,\infty)$$

is a solution of the differential equation

$$(\partial_t - \Delta_x)u = 0 \quad (x,t) \in \mathbb{R}^n \times (0,\infty)$$

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(a) We have for any $1 \leq i \leq 2$ and $x \in \mathbb{R}^n \backslash \{0\}$ (f is C^∞ in $\mathbb{R}^2 \backslash \{0\}$)

$$\partial_{x_i}^2 \log |x| = \frac{1}{2} \partial_{x_i}^2 \log |x|^2 = \partial_{x_i} \left(\frac{x_i}{|x|^2} \right) = \frac{1}{|x|^2} - \frac{2x_i^2}{|x|^4}$$

and

$$\Delta \log |x| = \frac{2}{|x|^2} - 2 \sum_{i=1}^{2} \frac{x_i^2}{|x|^4} = \frac{2}{|x|^2} - \frac{2}{|x|^2} = 0.$$

(b) For any $\alpha \in \mathbb{R}$ f_α is twice differentiable and

$$\partial_{x_i} f_\alpha(x) = \partial_{x_i} \left(\left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{\alpha}{2}} \right) = \frac{\alpha}{2} \cdot 2x_i \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{\alpha}{2}-1} = \alpha x_i |x|^{\alpha-2}$$

$$\partial_{x_i}^2 f_\alpha(x) = \alpha |x|^{\alpha-2} + \alpha(\alpha-2) x_i^2 |x|^{\alpha-4}$$

$$\Delta f_\alpha = \sum_{i=1}^{n} \alpha |x|^{\alpha-2} + \alpha(\alpha-2) x_i^2 |x|^{\alpha-4} = n\alpha |x|^{\alpha-2} + \alpha(\alpha-2)|x|^{\alpha-2} = \alpha(\alpha-n+2)|x|^{\alpha-4}.$$

Hence f_α is harmonic if and only if $\alpha = 0$ or $\alpha = 2-n$.

(c) For any $(x,t) \in \mathbb{R}^n \times (0,\infty)$

$$\partial_t u = -\frac{n}{2} t^{-\left(\frac{n}{2}+1\right)} e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4t^2} \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} = \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2} \right) u$$

$$\partial_{x_i} u = -\frac{2x_i}{4t} \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} = -\frac{x_i}{2t} u$$

$$\partial_{x_i}^2 u = -\frac{1}{2t} u + \frac{x_i^2}{4t^2} u = \left(-\frac{1}{2t} + \frac{x_i^2}{4t^2} \right) u$$

$$\Delta u = \sum_{i=1}^{n} \left(-\frac{1}{2t} + \frac{x_i^2}{4t^2} \right) u = \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2} \right) u = \partial_t u.$$

4.3. Continuity. Are the following statements true? If the statement is true, prove it. Otherwise give a counterexample.

(a) Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and p is a integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be function. If $g \circ f$ is continuous, then g is continuous or f is continuous.

(b) Let $X \subset \mathbb{R}^n$ be closed and p is a integer. If $f : X \to \mathbb{R}^p$ is a continuous function, then $f(X)$ is closed.

Solution:

(a) False. We give a counterexample in one dimension, i.e. $m = n = p = 1$. With the same idea, one can build counterexamples for any dimension. Let $X = [0, 2]$, $Y = [0, 3]$. $f(x) = x$ for $x \in [0, 1]$ and $f(x) = x + 1$ for $x \in (1, 2]$. $g(x) = x$ for $x \in [0, 1]$ and $g(x) = x - 1$ for $x \in (1, 3]$. Clearly, $(g \circ f)(x) = x$ is continuous but neither g nor f is continuous.

(b) False. Note that $\mathbb{R}^n$ is closed. We define $f(x) = \frac{1}{\|x\|^2+1}$, but $f(\mathbb{R}^n) = (0, 1]$ is not closed.

4.4. Partial derivative. Compute the derivatives of the following functions.

(a) $f : (0, \frac{\pi}{2}) \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = [\sin(y)]^x$;

(b) $f(x, y) = \frac{x-y}{x^2+y^2}$;

(c) $f(x, y) = x^2 y \sin(xy)$;

(d) $f(x, y, z) = xy^2z^3$.

Solution:

(a)

$$\frac{\partial f}{\partial x}(x, y) = e^{\log(\sin(y))x} \frac{\partial f}{\partial x}(\log(\sin(y))x) = e^{\log(\sin(y))x} \log(\sin(y)) = \sin(y)^x \log(\sin(y))$$
$$\frac{\partial f}{\partial y}(x, y) = e^{\log(\sin(y))x} \frac{\partial}{\partial y}(\log(\sin(y))x) = \sin(y)^x \frac{x\cos(y)}{\sin(y)} = x\cos(y)\sin(y)^{x-1}$$

(b)

$$\frac{\partial}{\partial x} f(x, y) = \frac{(x^2 + y^2) - (x - y)2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2},$$
$$\frac{\partial}{\partial y} f(x, y) = \frac{-(x^2 + y^2) - (x - y)2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}$$

(c)

$$\frac{\partial}{\partial x} f(x, y) = 2xy\sin(xy) + x^2y^2\cos(xy),$$
$$\frac{\partial}{\partial y} f(x, y) = x^2\sin(xy) + x^3y\cos(xy)$$

(d)

$$\frac{\partial}{\partial x} f(x, y, z) = y^2z^3,$$
$$\frac{\partial}{\partial y} f(x, y, z) = 2xyz^3,$$
$$\frac{\partial}{\partial z} f(x, y, z) = 3xy^2z^2$$