5.1. Directional derivative. Compute the directional derivatives of the following functions.

(a) $f(x,y) = \sin(x^2)\cos(y^2)$, $a = (0,2)$, $u = \frac{2}{\sqrt{5}}(\frac{1}{2}, 1)$;

(b) $f(x,y) = e^{-x}\log(y)$, $a = (0,1)$, $u = \frac{1}{\sqrt{17}}(-1,4)$;

(c) $f(x,y) = e^{y}\tan(x) + 4yx^3$, $a = (0,1)$, $u = \frac{1}{\sqrt{17}}(-1,4)$;

(d) $f(x,y) = x^5 y + \sin(\frac{x^2}{y})$, $a = (4,2)$, $u = \frac{1}{\sqrt{5}}(-1,-2)$.

Solution:

(a) From the definition, we have

$$D_u f(a) = \frac{d}{dt}\left(f(a+tu)\right)_{|t=0} \tag{1}$$

Then we compute

$$f(a+tu) = \sin\left(\frac{t^2}{5}\right)\cos\left((2+\frac{2t}{\sqrt{5}})^2\right)$$

Taking the derivative gives

$$D_u f(a) = \left[\cos\left(\frac{t^2}{5}\right)\frac{2t}{5}\cos\left(\left(2+\frac{2t}{\sqrt{5}}\right)^2\right) - \frac{4}{\sqrt{5}}\left(2+\frac{2t}{\sqrt{5}}\right)\sin\left(\frac{t^2}{5}\right)\sin\left(\left(2+\frac{2t}{\sqrt{5}}\right)^2\right)\right]_{t=0} = 0$$

(b) We have

$$f(a+tu) = e^{\frac{t}{\sqrt{17}}}\log\left(1+\frac{4t}{\sqrt{17}}\right).$$

Following the equation 1, we can compute

$$D_u f(a) = \left[\frac{e^{\frac{t}{\sqrt{17}}}}{\sqrt{17}}\log\left(1+\frac{4t}{\sqrt{17}}\right) + \frac{4e^{\frac{t}{\sqrt{17}}}}{\sqrt{17}(1+\frac{4t}{\sqrt{17}})}\right]_{t=0} = \frac{\log(1)+4}{\sqrt{17}} = \frac{4}{\sqrt{17}}.$$

(c) We have

$$f(a+tu) = e^{(1+\frac{4t}{\sqrt{17}})}\tan\left(\frac{-t}{\sqrt{17}}\right)+4\left(1+\frac{4t}{\sqrt{17}}\right)\left(\frac{-t}{\sqrt{17}}\right)^3 = e^{(1+\frac{4t}{\sqrt{17}})}\tan\left(\frac{-t}{\sqrt{17}}\right)-4\left(1+\frac{4t}{\sqrt{17}}\right)\left(\frac{t}{\sqrt{17}}\right)^3$$

Following the equation 1, we can compute

$$\begin{aligned} D_u f(a) = \Bigg[&\frac{4}{\sqrt{17}}e^{(1+\frac{4t}{\sqrt{17}})}\tan\left(\frac{-t}{\sqrt{17}}\right) + e^{(1+\frac{4t}{\sqrt{17}})}\frac{-1}{\cos^2(\frac{-t}{\sqrt{17}})\sqrt{17}} \\ &-\frac{16}{\sqrt{17}}\frac{t^3}{\sqrt{17}^3} - 4\left(1+\frac{4t}{\sqrt{17}}\right)\frac{3t^2}{\sqrt{17}^3}\Bigg]_{t=0} = \frac{-e}{\sqrt{17}} \end{aligned}$$

(d) We have

$$f(a+tu) = 2\left(4-\frac{t}{\sqrt{5}}\right)^5\left(1-\frac{t}{\sqrt{5}}\right) + \sin\left(\frac{(4-\frac{t}{\sqrt{5}})^2}{2(1-\frac{t}{\sqrt{5}})}\right),$$

and

$$\begin{aligned} D_u f(a) =& \Bigg[-\frac{10}{\sqrt{5}}\left(4-\frac{t}{\sqrt{5}}\right)^4\left(1-\frac{t}{\sqrt{5}}\right) - 2\left(4-\frac{t}{\sqrt{5}}\right)^5\frac{1}{\sqrt{5}} \\ &+\cos\left(\frac{(4-\frac{t}{\sqrt{5}})^2}{2(1-\frac{t}{\sqrt{5}})}\right)\left(-\frac{4-\frac{t}{\sqrt{5}}}{\sqrt{5}-t}+\frac{2(4-\frac{t}{\sqrt{5}})^2}{\sqrt{5}(2-\frac{2t}{\sqrt{5}})^2}\right)\Bigg]_{t=0} \\ =& -\frac{10}{\sqrt{5}}256 - 2\cdot 4^5\frac{1}{\sqrt{5}} + \cos(8)\left(-\frac{4}{\sqrt{5}}+\frac{2\cdot 16}{4\sqrt{5}}\right) \\ =& -\frac{2560}{\sqrt{5}} - \frac{2048}{\sqrt{5}} + \cos(8)\frac{4}{\sqrt{5}} \\ =& \frac{4608}{\sqrt{5}} + \cos(8)\frac{4}{\sqrt{5}}. \end{aligned}$$

5.2. Partial derivatives vs. differentiability. Define $f : \mathbb{R}^2 \to \mathbb{R}$ to be the following function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{wenn } (x,y) \neq (0,0) \\ 0 & \text{wenn } (x,y) = (0,0). \end{cases}$$

(a) Show that, for any point $a \in \mathbb{R}^2$ and any direction $u \in \mathbb{R}^2$, f admits a directional derivative $D_u f(a)$.

Tipp: To prove $D_u f(a)$ exists, one needs to show that $t \mapsto f(a+tu)$ is differentiable at the point $t = 0$. Recall the definition of differentiability in one variable and use the value $f(0,0)$.

(b) Show that, f is not differentiable at the point $(0,0)$.

Tipp: Recall that, when f is differentiable at the point $a \in \mathbb{R}^2$, f is also continuous at a.

Solution:

(a) Let $a = (x_0, y_0) \neq (0,0)$ and $u = (u_1, u_2) \neq (0,0)$, then we have

$$f(a+tu) = \frac{(x_0+tu_1)^2(y_0+tu_2)}{(x_0+tu_1)^4+(y_0+tu_2)^2}.$$

The denominator $\neq 0$ and bounded away from 0 from below. Then by composition rule, we have $t \mapsto f(a+tu)$ is differentiable at the point $t = 0$. Now we have showed that f admits all directional derivatives at $a \neq 0$. Next we prove, all directional derivatives at the point $a = (0,0)$ exist. First note that f vanishes on the line $\{(x,y) \in \mathbb{R}^2 \mid x = 0\}$. It follows that $\frac{\partial f}{\partial y}(0,0) = 0$. For any vector $u = (r\cos(\theta), r\sin(\theta))$ with $\sin(\theta) \neq 0$ we have

$$f((0,0)+tu) = f(tu) = \frac{t^3r^3\cos^2(\theta)\sin(\theta)}{t^4r^4\cos^4(\theta)+t^2r^2\sin^2(\theta)} = \frac{tr\cos^2(\theta)\sin(\theta)}{t^2r^2\cos^4(\theta)+\sin^2(\theta)}.$$

Again since $\sin^2(\theta) \neq 0$, we have $D_u f(0,0) = \frac{d}{dt}|_{t=0} f(tu) = \frac{r\cos^2(\theta)\sin(\theta)}{\sin^2(\theta)} = \frac{r\cos^2(\theta)}{\sin(\theta)}$. This concludes our proof.

(b) Note that f vanishes on the line $\{(x,y) \mid y=0,\ x\neq 0\}$. Then if the limit exists, we know $\lim_{(x,y)\to(0,0)} f(x,y)=0$. However, for $y=x^2$ we can see

$$f(x,x^2)=\frac{x^4}{x^4+x^4}=\frac{1}{2}.$$

The curve $\{(x,x^2) \mid x\in\mathbb{R}\}$ converges to $(0,0)$ as $x\to 0$, thus f is not continuous at $(0,0)$.

5.3. Jacobi matrix 1. Consider the functions

$$\alpha:\mathbb{R}^2\to\mathbb{R}^3\,,\ \ (x,y)\mapsto\begin{pmatrix} x^2+e^y \\ x+y \\ y\end{pmatrix}$$

and

$$\beta:\mathbb{R}^3\to\mathbb{R}^2\,,\ \ (u,v,w)\mapsto\begin{pmatrix} uv \\ w\end{pmatrix}.$$

Let $\gamma=\beta\circ\alpha$. Compute the Jacobi matrix of γ.

Solution: Since $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^k$ and $\mathbf{g}:\mathbb{R}^k\to\mathbb{R}^m$ are twice differentiable functions. The composition $\mathbf{g}\circ\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m$ is also differentiable and its derivatives at the point $p\in\mathbb{R}^n$ is given by

$$\left[\frac{\partial(\mathbf{g}\circ\mathbf{f})}{\partial\mathbf{x}}\right]_p=\left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right]_{\mathbf{f}(p)}\cdot\left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right]_p$$

The Jacobi matrix of α is

$$\begin{pmatrix} 2x & e^y \\ 1 & 1 \\ 0 & 1\end{pmatrix}$$

and the Jacobi matrix of β is

$$\begin{pmatrix} v & u & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

With the chain rule then we have

$$\begin{aligned}\left[\frac{\partial(\gamma_1,\gamma_2)}{\partial(x,y)}\right] &= \begin{pmatrix} v & u & 0 \\ 0 & 0 & 1\end{pmatrix}_{u=x^2+e^y,v=x+y,w=y}\cdot\begin{pmatrix} 2x & e^y \\ 1 & 1 \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} x+y & x^2+e^y & 0 \\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix} 2x & e^y \\ 1 & 1 \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} 3x^2+2xy+e^y & e^y(x+y+1)+x^2 \\ 0 & 1\end{pmatrix}.\end{aligned}$$

5.4. Jacobi matrix 2. Compute the Jacobi matrix of the function

$$f : \mathbb{R}^3 \to \mathbb{R}^3; \quad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\,\cos(\theta)\cos(\phi) \\ r\,\cos(\theta)\sin(\phi) \\ r\,\sin(\theta) \end{pmatrix}.$$

Solution:

$$\nabla f \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) & -r\sin(\theta)\cos(\phi) & -r\cos(\theta)\sin(\phi) \\ \cos(\theta)\sin(\phi) & -r\sin(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) \\ \sin(\theta) & r\cos(\theta) & 0 \end{bmatrix}.$$