6.1. Differentiation rules in more variables

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = \int_{\sin(x)}^{\cos(y)} e^{zt} dt.$$

Compute $\nabla f(\frac{\pi}{3}, \frac{\pi}{2}, 0)$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$ exist, and that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

Solution:

(a) Chain rule yields

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y, z) &= -\frac{\partial}{\partial x}\left(\int_{\cos(y)}^{\sin(x)} e^{zt} dt\right) = -e^{z\sin(x)}\cos(x) \\
\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y}\left(\int_{\sin(x)}^{\cos(y)} e^{zt} dt\right) = -e^{z\cos(y)}\sin(y) \\
\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z}\left(\int_{\sin(x)}^{\cos(y)} e^{zt} dt\right) = \int_{\sin(x)}^{\cos(y)} \frac{\partial}{\partial z}(e^{zt}) dt = \int_{\sin(x)}^{\cos(y)} te^{zt} dt
\end{aligned}$$

Let $(x, y, z) = (\frac{\pi}{3}, \frac{\pi}{2}, 0)$ and we have

$$\begin{aligned}
\frac{\partial f}{\partial x}(\tfrac{\pi}{3}, \tfrac{\pi}{2}, 0) &= -\cos(\tfrac{\pi}{3}) = -\tfrac{1}{2} \\
\frac{\partial f}{\partial y}(\tfrac{\pi}{3}, \tfrac{\pi}{2}, 0) &= -\sin(\tfrac{\pi}{2}) = -1 \\
\frac{\partial f}{\partial z}(\tfrac{\pi}{3}, \tfrac{\pi}{2}, 0) &= \int_{\sin(\frac{\pi}{3})}^{\cos(\frac{\pi}{2})} t dt = -\int_0^{\frac{\sqrt{3}}{2}} t dt = -\frac{3}{8}
\end{aligned}$$

Then

$$\nabla f(\tfrac{\pi}{3}, \tfrac{\pi}{2}, 0) = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{3}{8} \end{pmatrix},$$

(b) First we look at the function $y \mapsto \frac{\partial f}{\partial x}(0, y)$. For $y \neq 0$ we have

$$\frac{\partial f}{\partial x}(0, y) = \frac{\partial}{\partial x}\left(\frac{x^3 y - xy^3}{x^2 + y^2}\right)\bigg|_{x=0} = \left(\frac{3x^2 y - y^3}{x^2 + y^2} - \frac{2x(x^3 y - xy^3)}{(x^2 + y^2)^2}\right)\bigg|_{x=0} = -y.$$

To compute $\frac{\partial f}{\partial x}(0,0)$, note that

$$\frac{f(x,0)-f(0,0)}{x} = \frac{0-0}{x} = 0$$

for any $x \neq 0$. Then we can deduce $\frac{\partial f}{\partial x}(0,0) = 0$ and

$$\frac{\partial f}{\partial x}(0,y) = -y \quad \forall\, y \in \mathbb{R}$$

Obviously $y \mapsto \frac{\partial f}{\partial x}(0,y)$ is differentiable and $\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,y)\right)\Big|_{y=0} = -1$.

To compute $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ zu berechnen, first note that for any $x \neq 0$

$$\frac{\partial f}{\partial y}(x,0) = \frac{\partial}{\partial y}\left(\frac{x^3y-xy^3}{x^2+y^2}\right)\Bigg|_{y=0} = \left(\frac{x^3-3xy^2}{x^2+y^2} - \frac{2y(x^3y-xy^3)}{(x^2+y^2)^2}\right)\Bigg|_{y=0} = x$$

We also have

$$\frac{f(0,y)-f(0,0)}{y} = \frac{0-0}{y} = 0,$$

for any $y \neq 0$, which gives $\frac{\partial f}{\partial y}(0,0) = 0$. Now we can conclude

$$\frac{\partial f}{\partial y}(x,0) = x, \quad \forall\, x \in \mathbb{R}.$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,0)\right)\Bigg|_{x=0} = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

6.2. The geometry of the gradient Let $c \in \mathbb{R}$ be a constant and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant differentiable function. Assume that the equation $f(x,y) = c$ defines a curve C in the plane $\mathbb{R}^2$. I.e. there exists an interval $I \subset \mathbb{R}$ and an injective, differentiable map $\gamma : I \to \mathbb{R}^2$, so that

$$\gamma(I) = C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\} \tag{1}$$

and $\gamma'(t) \neq 0$ fr all $t \in I$. Prove the following statements:

(a) ∇f is perpendicular to C. I.e. for all $t \in I$ we have $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$.

(b) The directional derivative of f in a direction along C vanishes. I.e. $D_{\gamma'(t)} f(\gamma(t)) = 0$ for all $t \in I$.

(c) The directional derivative of f is largest in a direction perpendicular to C.

Solution:

(a) We know $I \ni t \mapsto f \circ \gamma(t)$ is a constant function, then $\frac{d}{dt} f \circ \gamma(t) = 0$. Chain rule yields

$$0 = \frac{d}{dt} f \circ \gamma(t) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t), \quad \forall\, t \in I.$$

This means ∇f is perpendicular to C.

(b) f is differentiable and with (a) we have

$$D_{\gamma'(t)}f(\gamma(t)) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = 0,$$

(c) Let $t \in I$ and $v \in \mathbb{R}^2$ with $\|v\| = 1$. Let n be a vector with $\|\mathsf{n}\| = 1$ and $\mathsf{n} \cdot \gamma'(t) = 0$. Also, n is a normal vector of C at the point $\gamma(t) \in C$. $\{\gamma'(t), \mathsf{n}\}$ forms a basis for $\mathbb{R}^2$ and there exist $a, b \in \mathbb{R}$ such that

$$v = a\gamma'(t) + b\mathsf{n}.$$

Since $\gamma'(t) \perp \mathsf{n}$, we have $1 = \|v\|^2 = |a|^2\|\gamma'(t)\|^2 + |b|^2\|\mathsf{n}\|^2 = |a|^2\|\gamma'(t)\|^2 + |b|^2$, then $|b| \leq 1$ with $|b| = 1$ if and only if $v = \pm\mathsf{n}$. Also $\nabla f(\gamma(t)) \perp \gamma'(t)$, we have

$$\begin{aligned} D_v f(\gamma(t)) &= \nabla f(\gamma(t)) \cdot v = \nabla f(\gamma(t)) \cdot (a\gamma'(t) + b\mathsf{n}) \\ &= b\nabla f(\gamma(t)) \cdot \mathsf{n} = bD_{\mathsf{n}} f(\gamma(t)) \leq \max\{D_{\mathsf{n}} f(\gamma(t)), D_{-\mathsf{n}} f(\gamma(t))\}. \end{aligned}$$

This concludes the proof.

6.3. Tangential planes Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \sin(x) - y^3 + y^2.$$

(a) Determine the equation of the tangential plane of the surface

$$\mathcal{G}(f) := \{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

at the point $(0, 3, f(0, 3)) = (0, 3, -18)$.

(b) Determine a constant $c \in \mathbb{R}$, so that the vector

$$\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$$

is perpendicular to the surface $\mathcal{G}(f)$ at the point $(\frac{\pi}{2}, 0, 1) \in \mathcal{G}(f)$.

Solution:

(a) First we compute

$$\frac{\partial f}{\partial x}(x, y) = \cos(x), \quad \frac{\partial f}{\partial y}(x, y) = -3y^2 + 2y \tag{2}$$

and then we have

$$\frac{\partial f}{\partial x}(0, 3) = 1, \quad \frac{\partial f}{\partial y}(0, 3) = -3 \cdot 9 + 2 \cdot 3 = 3(2 - 9) = -21.$$

A parametrization of $\mathcal{G}(f)$ is $F : \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(x, y) = (x, y, f(x, y))$$

Thus a basis of the tangent plane of $\mathcal{G}(f)$ at the point $(0, 3, -18)$ is

$$v := dF(0, 3)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ u := dF(0, 3)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We compute the differential $dF(0,3)$

$$dF(0,3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x}(0,3) & \frac{\partial f}{\partial y}(0,3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -21 \end{pmatrix}.$$

We also have

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u = \begin{pmatrix} 0 \\ 1 \\ -21 \end{pmatrix} \Rightarrow v \times u = \begin{pmatrix} -1 \\ 21 \\ 1 \end{pmatrix}$$

This means the point $(x, y, z) \in \mathbb{R}^3$ lies on the plane when

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -18 \end{pmatrix} + w$$

and $w \cdot (v \times u) = 0$. Then it is equivalent to

$$0 = (v \times u) \cdot \begin{pmatrix} x \\ y-3 \\ z+18 \end{pmatrix} = \begin{pmatrix} -1 \\ 21 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y-3 \\ z+18 \end{pmatrix} = -x + 21y - 3 \cdot 21 + z + 18.$$

Hence the tangent plane is given by

$$-x + 21y + z = 45.$$

(b) It is easy to see

$$\frac{\partial f}{\partial x}(\tfrac{\pi}{2}, 0) = 0, \quad \frac{\partial f}{\partial y}(\tfrac{\pi}{2}, 0) = 0$$

Similar to (a), the basis of the tangent plane of $\mathcal{G}(f)$ at $(\frac{\pi}{2}, 0, 1)$ is

$$v := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ u := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Also

$$v \times u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is perpendicular to $\mathcal{G}(f)$ at the point $(\frac{\pi}{2}, 0, 1)$, which gives $c = 0$.