7.1. Change of variable. Define $U \subset \mathbb{R}^2$ to be the open set

$$U = \mathbb{R}^2 \cap \left\{(x, y),\ y > 1, x > \max\left\{y, \frac{y}{y-1}\right\}\right\}.$$

1. Compute the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ in the new coordinate system of U

$$\begin{cases} u = xy \\ v = x + y \end{cases} \tag{1}$$

2. Let $f : U \to \mathbb{R}, (x, y) \mapsto e^{xy} \log(xy - x - y)$. Prove f is twice differentiable and compute Δf in U.

Solution:

1. We have $y = v - x$ and $u = xy = x(v - x)$, or

$$x^2 - vx + u = 0$$

Note that $x > y$, $2xy < x^2 + y^2$, then $v^2 > 4u$ and

$$x = \frac{1}{2}\left(v \pm \sqrt{v^2 - 4u}\right)$$

Thus

$$x = \frac{1}{2}\left(v + \sqrt{v^2 - 4u}\right), \quad y = v - x = \frac{1}{2}\left(v - \sqrt{v^2 - 4u}\right).$$

We then compute

$$\partial_x = \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = y\,\partial_u + \partial_v = \frac{1}{2}\left(v - \sqrt{v^2 - 4u}\right)\partial_u + \partial_v$$

and $(y\partial_u + \partial_v)y = 0$. Then we have

$$\partial_x^2 = y^2\partial_u^2 + 2y\partial_{uv}^2 + \partial_v^2 + (y\partial_u + \partial_v)y\,\partial_u = y^2\partial_u^2 + 2y\,\partial_{uv}^2 + \partial_v^2$$

and

$$\partial_y = x\,\partial_u + \partial_v = \frac{1}{2}\left(v + \sqrt{v^2 - 4u}\right)\partial_u + \partial_v$$
$$\partial_y^2 = x^2\partial_u^2 + 2x\,\partial_{uv}^2 + \partial_v^2$$

Since $(a - b)^2 + (a + b)^2 = 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\Delta u &= (x^2 + y^2)\partial_u^2 + 2(x + y)\partial_{uv}^2 + 2\,\partial_v^2 = ((x + y)^2 - 2xy)\partial_u^2 + 2\,v\partial_{uv}^2 + 2\,\partial_v^2 \\ &= (v^2 - 2u)\partial_u^2 + 2v\,\partial_{uv}^2 + 2\,\partial_v^2.\end{aligned}$$

(Computing x, y in terms of u, v may be helpful for intuition, but it is not needed for the aim of computing the Δ. For your intuition, I keep it in the solution.)

2. From the chain rule, the function f is twice differentiable, since in the domain $U \to \mathbb{R}_+^*$, the maps $(x,y) \mapsto xy - x - y$, $U \to \mathbb{R}, x \mapsto xy$, exp and log are smooth. From $f(u,v) = e^u \log(u-v)$, we can compute

$$\begin{aligned}
\partial_u f &= e^u \log(u-v) + \frac{e^u}{u-v} \\
\partial_u^2 f &= e^u \log(u-v) + 2\frac{e^u}{u-v} - \frac{e^u}{(u-v)^2} \\
\partial_{uv}^2 f &= -\frac{e^u}{u-v} + \frac{e^u}{(u-v)^2} \\
\partial_v f &= -\frac{e^u}{u-v} \\
\partial_v^2 f &= -\frac{e^u}{(u-v)^2}.
\end{aligned}$$

It yields

$$\begin{aligned}
\Delta f &= \left((v^2-2u)\log(u-v) + \frac{2(v^2-2u)}{u-v} - \frac{v^2-2u}{(u-v)^2} - \frac{2v}{u-v} + \frac{2v}{(u-v)^2} - \frac{2}{(u-v)^2}\right)e^u \\
&= \left(\left(x^2+y^2\right)\log(xy-x-y) + \frac{2(x^2+y^2-x-y)}{xy-x-y} + \frac{2(x+y-1)-(x^2+y^2)}{(xy-x-y)^2}\right)e^{xy}.
\end{aligned}$$

7.2. Second order derivatives. Compute the first and the second order derivatives of the following functions.

1. $f : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \cos(x)e^{-y}$.
2. Sei $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}, (x_1, \cdots, x_n) \mapsto \prod_{i=1}^n x_i$.

Solution:

1.

$$\begin{aligned}
&\partial_x f(x,y) = -\sin(x)e^{-y}, \quad \partial_{x,y}^2 f(x,y) = \sin(x)e^{-y} \quad \partial_x^2 f(x,y) = -\cos(x)e^{-y} \\
&\partial_y f(x,y) = -\cos(x)e^{-y}, \quad \partial_y^2 f(x,y) = \cos(x)e^{-y}.
\end{aligned}$$

2. For any $i \neq j \in \{1, \cdots, n\}$

$$\partial_{x_i} f(x) = \prod_{k\neq i} x_k, \quad \partial_{x_i}^2 f(x) = 0, \quad \partial_{x_i,x_j}^2 f(x) = \prod_{k\neq i,j} x_k.$$

7.3. Critical points.

Compute the critical points of the following functions.

1. $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x^3 + y^3 + 3xy$.
2. $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x^2 + y^2 - 2xy$.
3. $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto y(x-1)e^{-(x^2+y^2)}$.

Solution:

1. We know

$$\nabla f(x,y) = (3x^2 + 3y, 3y^2 + 3x) = 0$$

if and only if $x^2 = -y$ and $y^2 = -x$. Then $x^4 = y^2 = -x$, also $x(x^3+1) = 0$ and $y(y^3+1) = 0$. The real solution of the equation $X(X^3+1)$ are $X = 0, -1$, so the critical points of the system $x^2 = -y$, $y^2 = -x$ is given by

$$(0,0), \quad \text{and} \quad (x,y) = (-1,-1).$$

2. (x,x) is a critical point for any $x \in \mathbb{R}$. The computation is omitted since it is trivial.

3. We compute

$$\nabla f(x,y) = \left(y\,(1-2x(x-1))\,e^{-(x^2+y^2)}, (x-1)\left(1-2y^2\right)e^{-(x^2+y^2)}\right).$$

Hence the critical points of f are $x_0 = (1,0)$ and

$$x_1 = \left(\frac{1+\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \quad x_2 = \left(\frac{1+\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right), \quad x_3 = \left(\frac{1-\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), x_4 = \left(\frac{1-\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right).$$