8.1. Taylor polynomial. Compute the Taylor polynomials of the following functions.

1. $f : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \cos(x)e^{-y}$, at the point $(x_0, y_0) = (\pi/2, 0)$, up to the second order.
2. $f : \mathbb{R}^2 \backslash \{(x,y) : xy = 1\} \to \mathbb{R}, x \mapsto 1/(1-xy)$, at the point $(x_0, y_0) = (0,0)$, up to $2n-$th order $n \geq 1$.
3. $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \arctan(x^2 y)$, at the point $(x_0, y_0) = 0$, up to the second order.
4. $f : \mathbb{C} \to \mathbb{R}, z \mapsto \log(|z|^2 + 1)$, at the point $z = 0$, up to $2n-$th order $n \geq 1$.
5. Sei $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}, (x_1, \cdots, x_n) \mapsto \prod_{i=1}^n x_i$ at the point $x_0 = (2, \cdots, 2)$ up to the second order.

Solution:

1. Note that $f(0) = \cos\left(\frac{\pi}{2}\right) = 0$, and

$$\partial_x f(x,y) = -\sin(x)e^{-y}, \quad \partial^2_{x,y} f(x,y) = \sin(x)e^{-y} \quad \partial^2_x f(x,y) = -\cos(x)e^{-y}$$
$$\partial_y f(x,y) = -\cos(x)e^{-y}, \quad \partial^2_y f(x,y) = \cos(x)e^{-y}.$$

Then

$$\partial_x f(x_0,y_0) = -1, \quad \partial^2_{x,y} f(x_0,y_0) = 1, \quad f(x_0,y_0) = \partial^2_x f(x_0,y_0) = \partial_y f(x_0,y_0) = \partial^2_y f(x_0,y_0) = 0$$

and let $z = x - \pi/2$

$$T_2 f((x-\pi/2, y); (\pi/2, 0)) = T_2 f((z,y); (\pi/2, 0)) = -z + zy.$$

2. Note that

$$f(x,y) = \frac{1}{1-xy} = \sum_{n\in\mathbb{N}} (xy)^n.$$

Then for any $n \in \mathbb{N}$,

$$T_{2n} f((x,y); (0,0)) = 1 + \sum_{k=1}^{n} (xy)^k.$$

3. Note that $\arctan(t) = t + O(t^3)$ and

$$\arctan(x^2 y) = x^2 y + O(x^6 y^3).$$

Then $T_2 f((x,y), (0,0)) = 0$.

4. For any $|t| < 1$

$$\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k.$$

Then for any $|z| < 1$

$$\log(1+|z|^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} |z|^{2k}$$

and for any $w = w_1 + i w_2 \in \mathbb{C} \simeq \mathbb{R}^2$

$$T_{2n} f(w;0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} |w|^{2k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (w_1^2 + w_2^2)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{j=0}^{k} \binom{k}{j} w_1^{2j} w_2^{2(k-j)}.$$

5. Note that for any $i \neq j \in \{1, \cdots, n\}$

$$\partial_{x_i} f(x) = \prod_{k \neq i} x_k, \quad \partial^2_{x_i} f(x) = 0, \quad \partial^2_{x_i, x_j} f(x) = \prod_{k \neq i,j} x_k.$$

Then for any $1 \leq i \neq j \leq n$

$$f(2) = 2^n, \quad \partial_{x_i} f(2) = 2^{n-1}, \quad \partial^2_{x_i, x_j} f(2) = 2^{n-2},$$

Let $y = x - x_0, x_0 = (2, \ldots, 2)$ and we have

$$\begin{aligned} T_2 f(x - x_0; x_0) = T_2 f(y; x_0) &= f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} y^t \mathrm{Hess} f(x_0) y \\ &= 2^n + 2^{n-1} \sum_{i=1}^n y_i + 2^{n-2} \sum_{1 \leq i < j \leq n} y_i y_j. \end{aligned}$$

In the terms of order 2, we have a coefficient $\frac{1}{2}$ in Taylor formula, but we have two symmetric terms $\partial_i \partial_j$ and $\partial_j \partial_i$ for $1 \leq i < j \leq n$. We combine this 2 terms and end up with $(\frac{1}{2} + \frac{1}{2}) \partial^2_{x_i, x_j} f(2)$.

8.2. Critical points.

Compute local maximum and local minimum of the following functions.

1. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^3 + y^3 + 3xy$.
2. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2 - 2xy$.
3. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto y(x - 1)e^{-(x^2+y^2)}$.
4. $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 + z^2 + 2xyz$.

Note that in this solution, we include some information about global maximum and global minimum. Let $f : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$. We say f has a global maximum at $x_0 \in \Omega$ if

$$f(x) \leq f(x_0) \quad \text{for any } x \in \Omega.$$

Similarly, we say f has a global minimum at $x_0 \in \Omega$ if

$$f(x) \geq f(x_0) \quad \text{for any } x \in \Omega.$$

Solution:

1. We know

$$\nabla f(x, y) = (3x^2 + 3y, 3y^2 + 3x) = 0$$

if and only if $x^2 = -y$ and $y^2 = -x$. Then $x^4 = y^2 = -x$, also $x(x^3 + 1) = 0$ and $y(y^3 + 1) = 0$. The real solution of the equation $X(X^3 + 1)$ are $X = 0, -1$, so the solution of the system $x^2 = -y$, $y^2 = -x$ is given by

$$(0, 0), \quad \text{and} \quad (x, y) = (-1, -1).$$

We have

$$\mathrm{Hess}\, f(x, y) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$$

also

$$\operatorname{Hess} f(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \operatorname{Hess} f(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$$

Note that $(0,0)$ is neither local minimum nor local maximum, since $\operatorname{Hess} f(0,0)$ has a positive eigenvalue 3 and a negative eigenvalue -3. $(-1,-1)$ is a local maximum, since $\operatorname{Hess} f(-1,1)$ has two negative eigenvalues -3 and -9.

2. Note that $f(x,y) = (x-y)^2 \geq 0$, also $f(x,0) = x^2 \to \infty$ as $x \to \pm\infty$ and f have no global maximum. For any $x \in \mathbb{R}$, $f(x,x) = 0 = \min_{\mathbb{R}^2} f$, then (x,x) is a global minimum for any $x \in \mathbb{R}$, then it is also a local minimum. Remark that one can check these critical points are not non-degenerate critical points.

3. Note that

$$0 \leq |f(x,y)| \leq \frac{1}{2}\left(y^2 + (x-1)^2\right) e^{-(x^2+y^2)} \to 0 \text{ as } |(x,y)| \to \infty,$$

and $f(x,0) = f(1,y) = 0$ for any $x, y \in \mathbb{R}$ and f has global maximum or minimum. We compute

$$\nabla f(x,y) = \left(y\,(1 - 2x(x-1))\, e^{-(x^2+y^2)}, (x-1)\left(1-2y^2\right) e^{-(x^2+y^2)}\right).$$

Hence the critical points of f are $x_0 = (1,0)$ and

$$x_1 = \left(\frac{1+\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \quad x_2 = \left(\frac{1+\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right), \quad x_3 = \left(\frac{1-\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), x_4 = \left(\frac{1-\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right).$$

We have

$$\partial_x^2 f(x,y) = y\,(2(1-2x) - 2x(1-2x(x-1)))\, e^{-(x^2+y^2)}, \quad \partial_{xy}^2 f(x,y) = (1-2y^2)(1-2x(x-1))e^{-(x^2+y^2)}$$
$$\partial_y^2 f(x,y) = (x-1)\left(-4y - 2y(1-2y^2)\right) e^{-(x^2+y^2)} = -2y(x-1)(3-2y^2)e^{-(x^2+y^2)}.$$

and

$$\operatorname{Hess} f(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-1},$$

and $(1,0)$ is neither local maximum nor local minimum (we have $f(1,0) = 0$, but $f(1+\epsilon,\epsilon) = \epsilon^2 e^{-(1+\epsilon)^2-\epsilon^2} > 0$ and $f(1+\epsilon,-\epsilon) = -\epsilon^2 e^{-(1+\epsilon)^2-\epsilon^2} < 0$ for any $\epsilon > 0$). When $2x^2 - 2x - 1 = 0$ and $1 - 2y^2 = 0$, we have

$$\partial_x^2 f(x,y) = 2y(1-2x)e^{-(x^2+y^2)} \neq 0, \quad \partial_{xy}^2 f(x,y) = 0$$
$$\partial_y^2 f(x,y) = -4y(x-1)e^{-(x^2+y^2)} \neq 0.$$

Note that x_i (for $1 \leq i \leq 4$) a local maximum if and only if $\partial_x^2 f(x_i), \partial_y^2 f(x_i) < 0$, x_i is a local minimum if and only if $\partial_x^2 f(x_i) > 0$, $\partial_y^2 f(x) > 0$, and x_i is neither local maximum nor local minimum when $\partial_x^2 f(x_i)\partial_y^2 f(x_i) < 0$. Then $x_i = (x,y)$ is a global minimum, when

$$2y(1-2x) > 0 \text{ and } -4y(x-1) > 0 \iff \min\{y(1-2x), y(1-x) > 0\}.$$

Then x_2, x_3 are local minimum and x_1, x_4 are local maximum. Then

$$\begin{aligned}
f(x_1) &= \frac{\sqrt{3}-1}{2\sqrt{2}} e^{-\frac{3+\sqrt{3}}{2}} = 0,02\cdots \\
f(x_4) &= \frac{(1+\sqrt{3})}{2\sqrt{2}} e^{-\frac{3-\sqrt{3}}{2}} = 0,51\cdots \\
f(x_2) &= -\frac{1+\sqrt{3}}{2\sqrt{2}} e^{-\frac{3-\sqrt{3}}{2}} = -0,51\cdots \\
f(x_3) &= -\frac{\sqrt{3}-1}{2\sqrt{2}} e^{-\frac{3+\sqrt{3}}{2}} = -0.02\cdots
\end{aligned}$$

x_4 is a global maximum and x_2 is a global minimum.

4. Compute the gradient

$$\nabla f(x,y,z) = (2x + 2yz, 2y + 2xz, 2z + 2xy).$$

and $\nabla f(x,y,z) = 0$ implies

$$2x + 2yz = 0, \qquad 2y + 2xz = 0, \qquad 2z + 2xy = 0.$$

Then all the critical points are given by

$$\operatorname{Krit}(f) = \{(0,0,0), (-1,1,1), (1,-1,1), (1,1,-1), (-1,-1,-1)\}.$$

The Hessian matrix of f is given by

$$H(f,(x,y,z)) = \begin{pmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{pmatrix}.$$

Hence the Hessian matrix at the point 0 is given

$$H(f,(0,0,0)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is a postive Diagonal matrix and positive-definite, so $(0,0,0)$ is a local minimum.

The matrices

$$H(f,(-1,1,1)) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}, \qquad H(f,(1,-1,1)) = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

$$H(f,(1,1,-1)) = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \qquad H(f,(-1,-1,-1)) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

have the same determinant -32. We can easily see they are indefinite. Indeed, negative determinant means that their three eigenvalues are nonzero and the product of the eigenvalues is negative. However, the three eigenvalues cannot be all negative since the their traces are positive, so the only possibility is that they have 2 positive eigenvalues and 1 negative eigenvalue. Therefore, $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$ are saddle points.