9.1. Differentiability The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) := \begin{cases} 0 & \text{für } (x,y) = 0 \\ \frac{x^3}{\sqrt{x^2+y^2}} & \text{für } (x,y) \neq 0 \end{cases}$$

Prove that f is differentiable.

Solution: It is obvious that f is differentiable at any point $(x,y) \neq (0,0)$. At the point $(0,0)$, we first show that the partial derivatives exist

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^2-0}{h} = \lim_{h\to 0} h = 0$$
$$\partial_y f(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Now we know the partial derivatives exist. If f is differentiable at the point $(0,0)$, the differential of f at the point $(0,0)$ is given by the linear map

$$\begin{aligned} \nabla f(0,0) : \mathbb{R}^2 &\to \mathbb{R} \\ v &\mapsto \langle \nabla f(0,0), v\rangle = \partial_x f(0,0)\cdot v_1 + \partial_y f(0,0)\cdot v_2 = 0\cdot v_1 + 0 \cdot v_2 \end{aligned}$$

This is $\nabla f(0,0) = 0$.

Now we prove that indeed the map $\nabla f(0,0) = 0$ is the the differential of f at the point $(0,0)$.

$$\begin{aligned} &\lim_{(v_1,v_2)\to(0,0)} \frac{f(v_1,v_2)-f(0,0)-\nabla f(0,0)(v_1,v_2)}{||(v_1,v_2)||} \\ &\qquad = \lim_{(v_1,v_2)\to(0,0)} \frac{v_1^3}{v_1^2+v_2^2} = 0. \end{aligned}$$

Note the fact that $\left|\frac{v_1^3}{v_1^2+v_2^2}\right| \leq |v_1|$ for any $v_1 \neq 0$. This concludes our proof.

9.2. Tangential plane Given the function

$$\begin{aligned} f\colon \Omega &\to \mathbb{R} \\ (x,y) &\mapsto \sqrt{1-x^2-y^2}, \end{aligned}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$, compute the tangential plane of the graph of f at the point $(x_0, y_0, f(x_0,y_0))$.

Solution: We compute

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}}$$
$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

and then the tangential plane is given by

$$\begin{aligned} z &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) \\ &= \sqrt{1-x_0^2-y_0^2} - \frac{x_0}{\sqrt{1-x_0^2-y_0^2}}(x-x_0) - \frac{y_0}{\sqrt{1-x_0^2-y_0^2}}(y-y_0). \end{aligned}$$

9.3. Hessian matrix Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = e^{xy}(\sin(x) + 3\cos(xy))$$

Compute the Hessian matrix of f at the point $(x,y) = (0,1)$.

Solution: With the product rule, we have

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= ye^{xy}(\sin(x) + 3\cos(xy)) + e^{xy}(\cos(x) - 3y\sin(xy))\\ \frac{\partial f}{\partial y}(x,y) &= xe^{xy}(\sin(x) + 3\cos(xy)) - 3e^{xy}x\sin(xy).\end{aligned}$$

Differentiating them once again gives

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial x}(x,y) =& y^2e^{xy}(\sin(x) + 3\cos(xy)) + ye^{xy}(\cos(x) - 3y\sin(xy))\\ &+ ye^{xy}(\cos(x) - 3y\sin(xy)) + e^{xy}(-\sin(x) - 3y^2\cos(xy))\\ \frac{\partial^2 f}{\partial y \partial x}(x,y) =& \frac{\partial^2 f}{\partial x \partial y}(x,y) = (e^{xy} + xye^{xy})(\sin(x) + 3\cos(xy)) - 3yxe^{xy}\sin(xy)\\ &+ xe^{xy}(\cos(x) - 3y\sin(xy)) - 3e^{xy}(\sin(xy) + xy\cos(xy))\\ \frac{\partial^2 f}{\partial y \partial y}(x,y) =& x^2e^{xy}(\sin(x) + 3\cos(xy)) - 3x^2e^{xy}\sin(xy) - 3e^{xy}x^2(\sin(xy) + \cos(xy))\end{aligned}$$

At the point $(x,y) = (0,1)$, we have

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial x}(0,1) =& 2\\ \frac{\partial^2 f}{\partial y \partial x}(0,1) =& \frac{\partial^2 f}{\partial x \partial y}(0,1) = 3\\ \frac{\partial^2 f}{\partial y \partial y}(0,1) =& 0.\end{aligned}$$

Then we can compute Hessian matrix

$$\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix}\right|_{(x,y)=(0,1)} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}.$$

9.4. Line integrals Compute the following line integrals.

1. $v(x,y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$, from $(-1,1)$ to $(1,1)$ along the curve $y = x^2$.
2. $v(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, from $(0,0)$ to $(2,0)$ along the curve $y = 1 - |1 - x|$.
3. $v(x,y,z) = \begin{pmatrix} x \\ y \\ xz - y \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} t^2 \\ 2t \\ 4t^3 \end{pmatrix}$, $t \in [0,1]$.
4. $v(x,y) = \begin{pmatrix} 2a - y \\ x \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} a(t - \sin(t)) \\ a(1 - \cos(t)) \end{pmatrix}$, $t \in [0, 2\pi]$, with a constant $a \in \mathbb{R}$.

Solution:

1. A parametrization of the curve is given by $\gamma : [-1,1] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \Rightarrow \gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix},$$

then we have

$$v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = t^2 - 2t^3 + 2t^5 - 4t^4.$$

Now we can compute

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_{-1}^{1} t^2 - 2t^3 + 2t^5 - 4t^4 \; dt = \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5} \right]_{t=-1}^{1} \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left(\frac{-1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{4}{5} \right) \\ &= \frac{1}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{5} = \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}, \end{aligned}$$

2. A parametrization of the curve is given by $\gamma(t) = (t, \gamma_2(t))$, $t \in [0,2]$, with

$$\gamma_2(t) = \begin{cases} t, & t \in [0,1] \\ 2-t, & t \in [1,2]. \end{cases}$$

then we have for $t \in [0,1]$

$$v(\gamma(t)) = \begin{pmatrix} 2t^2 \\ 0 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^2$$

and for $t \in [1,2]$

$$v(\gamma(t)) = \begin{pmatrix} t^2 + (2-t)^2 \\ t^2 - (2-t)^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2(2-t)^2$$

Now we can compute

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_0^1 2t^2 \; dt + \int_1^2 2(2-t)^2 dt = \left[\frac{2t^3}{3} \right]_{t=0}^{1} + 2 \left[\frac{-(2-t)^3}{3} \right]_{t=1}^{2} \\ &= \frac{2}{3} + 2\left(\frac{1}{3}\right) = \frac{4}{3}, \end{aligned}$$

3. We have

$$v(\gamma(t)) = \begin{pmatrix} t^2 \\ 2t \\ 4t^5 - 2t \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} 2t \\ 2 \\ 12t^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^3 + 4t + 12t^2(4t^5 - 2t),$$

and this leads to

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_0^1 2t^3 + 4t + 12t^2(4t^5 - 2t) \; dt = \int_0^1 4t + 48t^7 - 22t^3 \; dt \\ &= \left[2t^2 + 6t^8 - \frac{22t^4}{4} \right]_{t=0}^{1} = 2 + 6 - \frac{11}{2} = \frac{4 + 12 - 11}{2} = \frac{5}{2}. \end{aligned}$$

4. We have

$$v(\gamma(t)) = \begin{pmatrix} 2a - a(1-\cos(t)) \\ a(t - \sin(t)) \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} a(1-\cos(t)) \\ a\sin(t) \end{pmatrix}$$

and we have

$$\begin{aligned} v(\gamma(t)) \cdot \gamma'(t) =& 2a^2 - 2a^2\cos(t) - a^2 + a^2\cos(t) + a^2\cos(t) \\ & - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t). \end{aligned}$$

Now we can compute

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_0^{2\pi} 2a^2 - 2a^2\cos(t) - a^2 + a^2\cos(t) + a^2\cos(t) \\ &\qquad - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t) \; dt \\ &= \int_0^{2\pi} a^2 - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t) \; dt = -2\pi a^2. \end{aligned}$$