10.1. Conservative vector-fields

For which of the following vector-fields v does there exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $v = \nabla f$?

1. $v(x, y) = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$
2. $v(x, y) = \begin{pmatrix} x^2 - y \\ x^3 + 2xy \end{pmatrix}$
3. $v(x, y) = \begin{pmatrix} x^3 + 2xy \\ x^2 - y \end{pmatrix}$
4. $v(x, y) = \begin{pmatrix} x^3 - xy^2 \\ x^2 y - y^5 \end{pmatrix}$

Solution : As $\mathbb{R}^2$ is star-shaped, it suffices to check the condition $\partial_y v_1 = \partial_x v_2$, where $v = (v_1, v_2)$ to determine if v is conservative or not. Indeed, for star-shaped domains, $\partial_y v_1 = \partial_x v_2$ implies that v is conservative, from Theorem 4.1.17. For general open domains, v is conservative implies $\partial_y v_1 = \partial_x v_2$ from Proposition 4.1.13.

1. We have $\partial_y v_1 = -1 \neq 1 = \partial_x v_2$, so v is not conservative.
2. We have $\partial_y v_1(x, y) = -1$, while $\partial_x v_2(x, y) = 3x^2$. Therefore, v is not conservative.
3. We have $\partial_y v_1(x, y) = 2x = \partial_x v_2(x, y)$, so v is conservative.
4. We have $\partial_y v(x, y) = -2xy$ while $\partial_x v(x, y) = 2xy$. Therefore, v is not conservative.

10.2. An example Let $V : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2$ be defined by

$$V(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

1. Show that V satisfies the necessary condition to be conservative in proposition 4.1.13.
2. Compute the pathintegral of V around the curve $\gamma(t) = (\sin(t), -\cos(t))$, $0 \leq t \leq 2\pi$.

Solution:

1. Writing $V = (V_1, V_2)$, we have
$$\partial_y V_1(x, y) = \partial_x V_2(x, y) = -\frac{2xy}{x^2 + y^2}.$$
2. We have
$$\int_0^{2\pi} V(s) \cdot d\vec{s} = \int_0^{2\pi} (\sin(t) \cos(t) - \cos(t) \sin(t))\, dt = 0$$

10.3. Pathintegrals Compute in the following exercises the pathintegral of the vectorfield v along the path.

1. $v(x, y) = \left(\dfrac{x}{x^2 + y^2 + 1}, \dfrac{y}{x^2 + y^2 + 1} \right)$, along the circle $x^2 + y^2 - 2x = 1$.

2. $v(x, y, z) = \begin{pmatrix} 2xy^2z \\ 2x^2yz \\ x^2y^2 - 2z \end{pmatrix}$, along the path $\gamma(t) = \left(\cos(t), \frac{\sqrt{3}}{2}\sin(t), \frac{1}{2}\sin(t)\right)$, $0 \leq t \leq 2\pi$.

Solution:

1. We see by direct computation that

$$\partial_y v_1 = \frac{-2xy}{(x^2 + y^2 + 1)^2} = \partial_x v_2$$

and we know $\mathbb{R}^2$ is star-shaped. Therefore, v is conservative and the integral evaluates to 0. We can also verify this by computing g. Note that $v(x, y) = \nabla g(x, y)$. We have

$$\partial_x g = \frac{x}{x^2 + y^2 + 1} \quad \Rightarrow \quad g = \int \frac{x}{x^2 + y^2 + 1} dx = \log(\sqrt{x^2 + y^2 + 1}) + C(y)$$

and

$$\partial_y g = \frac{y}{x^2 + y^2 + 1} \overset{!}{=} \frac{y}{x^2 + y^2 + 1} + \partial_y C(y) \quad \Rightarrow \quad C(y) = 0$$

We have $g = \log(\sqrt{x^2 + y^2 + 1})$ and the integral is $\int_\gamma v d\vec{s} = g(\gamma(2\pi)) - g(\gamma(0)) = 0$

2. We can either check the condition on the first derivatives

$$\partial_y v_1 = 4xyz = \partial_x v_2, \quad \partial_z v_1 = 2xy^2 = \partial_x v_3, \quad \partial_z v_2 = 2x^2y = \partial_y v_3$$

and see v is conservative, or we can find g to verify that v is conservative:

$$\begin{aligned} \partial_x g = 2xy^2z \quad &\Rightarrow \quad g = \int 2xy^2z dx = x^2y^2z + C(y, z) \\ \partial_y g = 2x^2yz \overset{!}{=} 2x^2yz + \partial_y C(y, z) \quad &\Rightarrow \quad C(y, z) = C(z) \\ \partial_z g = x^2y^2 - 2z \overset{!}{=} x^2y^2 + \partial_z C(z) \quad &\Rightarrow \quad C(z) = \int -2z dz = -z^2 \\ v(x, y, z) = \nabla g &= \nabla(x^2y^2z - z^2) \end{aligned}$$

Since v is conservative, the integral vanishes again: $\int_\gamma v d\vec{s} = g(\gamma(2\pi)) - g(\gamma(0)) = 0$