11.1. Fubini's theorem for explicit functions I

(1) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4y - y^5x + y^3\right) dxdy$$

(2) Let $D^2 = \mathbb{R}^2 \cap \{(x,y) : x^2 + y^2 \leq 1\}$ be the unit disk in the plan. Compute

$$\int_{D^2} x^2y^2 dxdy$$

by following the following steps.

(a) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \frac{\pi}{32}$$

(b) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x,y)dxdy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dy\right) dx.$$

(c) Compute

$$\int_{D^2} x^2y^2 dxdy,$$

by making the formula of question (2) and a (1-dimensional) change of variable using trigonometric functions and symmetry.

Solution:

(1) We have

$$\begin{aligned}
\int_{[-1,1]\times[2,3]} \left(x^4y - y^5x + y^3\right) dxdy &= \int_2^3 \left(\int_{-1}^1 x^4y - y^5x + y^3dx\right) dy \\
&= \int_2^3 \left(\left[y\frac{x^5}{5} - y^5\frac{x^2}{2}\right]_{-1}^1 + 2y^3\right) dxdy \\
&= \int_2^3 \left(\frac{2y}{5} + 2y^3\right) dy \\
&= \left[\frac{y^2}{5} + \frac{y^4}{2}\right]_2^3 = \frac{9-4}{5} + \frac{81-16}{2} = \frac{67}{2}.
\end{aligned}$$

(2) (a) Using $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\cos^2 + \sin^2 = 1$, we obtain

$$\begin{aligned}
\cos^4(\theta)\sin^2(\theta) &= \frac{1}{4}\cos^2(\theta)\sin^2(2\theta) = \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(1-\cos^2(2\theta)\right) \\
&= \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(4\theta)}{2}\right) \qquad (1) \\
&= \frac{1}{16}\left(1+\cos(2\theta)-\cos(4\theta)-\cos(2\theta)\cos(4\theta)\right) \\
&= \frac{1}{32}\left(2+2\cos(2\theta)-2\cos(4\theta)-2\cos(2\theta)\cos(4\theta)\right) \\
&= \frac{1}{32}\left(2+2\cos(2\theta)-2\cos(4\theta)-\cos(2\theta)-\cos(6\theta)\right) \\
&= \frac{1}{32}\left(2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)\right) \qquad (2)
\end{aligned}$$

where we used

$$\cos(2\theta)\cos(4\theta) = \frac{1}{2}\left(\cos(2\theta)+\cos(6\theta)\right),$$

an identity which can be derived from the de Moivre's formula

$$\begin{aligned}
\cos(2\theta)\cos(4\theta) &= \frac{\left(e^{2i\theta}+e^{-2i\theta}\right)}{2}\frac{\left(e^{4i\theta}+e^{-4i\theta}\right)}{2} = \frac{1}{4}\left(e^{2i\theta}+e^{-2i\theta}+e^{6i\theta}+e^{-6i\theta}\right) \\
&= \frac{1}{2}\left(\cos(2\theta)+\cos(6\theta)\right).
\end{aligned}$$

Now, we have obviously for all integer $k \geq 1$

$$\int_0^{\frac{\pi}{2}} \cos(2k\theta)d\theta = \left[\frac{\sin(2k\theta)}{k}\right]_0^{\frac{\pi}{2}} = \sin(\pi k) = 0. \qquad (3)$$

Therefore, by (1) and (3), we obtain

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{32}\left(2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)\right)d\theta = \frac{\pi}{32}.$$

Remark: One could also directly expand $\cos^4(\theta)\sin^2(\theta)$ with de Moivre's formula, but the computation would be slightly longer.

(b) We have for all $(x, y) \in D^2$ the inequality

$$x^2 + y^2 \leq 1 \qquad (4)$$

which implies that $-1 \leq x \leq 1$. Therefore, (4) holds if and only $-1 \leq x \leq 1$ and

$$y^2 \leq 1 - x^2$$

which is equivalent to (notice that $1 - x^2 \geq 0$) $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Finally, we have proved that

$$\begin{aligned}
D^2 &= \mathbb{R}^2 \cap \{(x,y) : x^2+y^2 \leq 1\} \\
&= \mathbb{R}^2 \cap \{(x,y) : -1 \leq x \leq 1 \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}. \qquad (5)
\end{aligned}$$

The integral formula is then a direct consequence of (5) and Fubini's theorem.

(c) Using the formula in (b), we find

$$\begin{aligned}\int_{D^2} x^2y^2dxdy &= \int_{-1}^{1} x^2\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2dy\right)dx = \int_{-1}^{1} x^2\left[\frac{y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \frac{2}{3}\int_{-1}^{1} x^2\left(1-x^2\right)^{\frac{3}{2}}dx = \frac{4}{3}\int_{0}^{1} x^2\left(1-x^2\right)^{\frac{3}{2}}dx \end{aligned} \tag{6}$$

where we used the symmetry of the integral in the last equality (formally, one can make a change of variable $t=-x$ in the integral $\int_{-1}^{0}$ to obtain the result). Now, we make the change of variable $x=\sin(\theta)$ to obtain (using $1-\sin^2=\cos^2$)

$$\int_0^1 x^2\left(1-x^2\right)^{\frac{3}{2}}dx = \int_0^{\frac{\pi}{2}} \sin^2(\theta)\left(1-\sin^2(\theta)\right)^{\frac{3}{2}}\cos(\theta)d\theta = \int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^4(\theta)d\theta. \tag{7}$$

Therefore, thanks of the computation in (a), (6) and (7)

$$\int_{D^2} x^2y^2dxdy = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^4(\theta)d\theta = \frac{4}{3}\times\frac{\pi}{32} = \frac{\pi}{24}.$$

Remark: Once we know the change of variables, we can use polar coordinates to find

$$\begin{aligned}\int_{D^2} x^2y^2dxdy &= \int_0^1\int_0^{2\pi} r^5\cos^2(\theta)\sin^2(\theta)d\theta dr = \frac{1}{6}\int_0^{2\pi}\cos^2(\theta)\sin^2(\theta)d\theta \\ &= \frac{1}{24}\int_0^{2\pi}\sin^2(2\theta)d\theta = \frac{1}{24}\int_0^{2\pi}(1-\cos^2(2\theta))d\theta \\ &= \frac{1}{24}\int_0^{2\pi}\left(1-\frac{1+\cos(4\theta)}{2}\right)d\theta = \frac{\pi}{24},\end{aligned}$$

where we used $\sin(2\theta)=\cos(\theta)\sin(\theta)$ and $\cos(2\theta)=2\cos^2(\theta)-1$.

11.2. Fubini's theorem for explicit functions II

Compute the following double integrals $\int_D f(x,y)dxdy$, where the continuous function $f: D\to\mathbb{R}$ and the domain D are given by

1. $f(x,y)=x$, and $D=\mathbb{R}^2\cap\{(x,y): y\geq 0, x-y+1\geq 0, x+2y-4\leq 0\}$.
2. $f(x,y)=\cos(xy)$, and $D=\mathbb{R}^2\cap\{(x,y): 1\leq x\leq 2, 0\leq xy\leq\frac{\pi}{2}\}$.
3. $f(x,y)=\frac{1}{(x+y)^3}$, and $D=\mathbb{R}^2\cap\{(x,y): 1\leq x\leq 3, y\geq 2, x+y\leq 5\}$.
4. $f(x,y)=\frac{xy}{1+x^2+y^2}$, and $D=\mathbb{R}^2\cap\{(x,y): 0\leq x\leq 1,\ 0\leq y\leq 1,\ x^2+y^2\geq 1\}$.

Solution:

1. For all $(x,y)\in D$, we have $y\geq 0$, and $y-1\leq x\leq 4-2y$, which is non-empty if and only $y-1\leq 4-2y$, or $y\leq\dfrac{5}{3}$. Therefore, we have

$$\int_D f(x,y)dxdy = \int_0^{\frac{5}{3}}\int_{y-1}^{4-2y} xdxdy = \frac{1}{2}\int_0^{\frac{5}{3}}\left((4-2y)^2-(y-1)^2\right)dy = \frac{275}{54}.$$

2. We have

$$\int_D f(x,y)dxdy = \int_1^2 \left(\int_0^{\frac{\pi}{2x}} \cos(xy)dy \right) dx = \int_1^2 \left[\frac{\sin(xy)}{x} \right]_0^{\frac{\pi}{2x}} dx = \int_1^2 \frac{dx}{x} = \log(2).$$

3. We have

$$\int_D f(x,y)dxdy = \int_1^3 \int_2^{5-x} \frac{dy}{(x+y)^3} dx = \int_1^3 -\frac{1}{2} \left(\frac{1}{25} - \frac{1}{(x+2)^2} \right) dx = \frac{2}{75}.$$

4. We have

$$\begin{aligned} \int_D f(x,y)dxdy &= \int_0^1 \left(\int_{\sqrt{1-x^2}}^1 \frac{xy}{1+x^2+y^2} dy \right) dx \\ &= \int_0^1 \left[\frac{x}{2} \log\left(1+x^2+y^2\right) \right]_{\sqrt{1-x^2}}^1 dx \\ &= \int_0^1 \frac{x}{2} \left(\log(2+x^2) - \log(2) \right) dx \\ &= \frac{3}{4} \log\left(\frac{3}{2}\right) - \frac{1}{4}. \end{aligned}$$

11.3. Fubini's theorem for explicit functions III

Compute the area of the domain

$$D = \mathbb{R}^2 \cap \{(x,y) : -1 \leq x \leq 1, x^2 \leq y \leq 4 - x^3\}.$$

Solution: We have

$$\text{area}(D) = \int_D dxdy = \int_{-1}^1 \int_{x^2}^{4-x^3} dydx = \int_{-1}^1 \left(4 - x^3 - x^2\right) dx = \left[4x - \frac{x^4}{4} - \frac{x^3}{3} \right]_{-1}^1 = \frac{22}{3}.$$

Remark: Notice that for all $-1 \leq x \leq 1$, we have $4 - x^3 \geq 3 > x^2$, so the decomposition of the domain in the previous question is correct.

11.4. Are the following sets negligible in $\mathbb{R}^3$?

1. $\{(i,j,k) \in \mathbb{R}^3 \mid i,j,k \in \mathbb{Z}, i^2 + j^2 + k^2 < 2019\}$.
2. $\{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1, x, y \in [0,1]\}$.

Solution:

1. Negligible. This set has finitely many distinct points. We just need to build finitely many constant maps from $[0,1]$ to these finitely many distinct points.
2. Negligible. We just need one map from $[0,1] \times [0,1]$ to $\mathbb{R}^3$, which is gvien by $(x,y) \to (x, y, 1 - x - y)$.