13.1. Change of variables. Let $D = \mathbb{R}^2 \cap \{(x,y) : x^2+y^2-2x \leq 0\}$. Compute

$$\int_D \sqrt{x^2+y^2}dxdy$$

Hint: Notice that D is a disk.

Solution: First, we have

$$x^2+y^2-2x = (x-1)^2+y^2-1,$$

so D is the disk as centre $(1,0)$ and radius 1. By using polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, we find

$$(x-1)^2+y^2 = r^2-2r\cos(\theta) \leq 0 \Leftrightarrow 0 \leq r \leq 2\cos(\theta),$$

which implies in particular that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, by the change of variables formula, we have

$$\int_D \sqrt{x^2+y^2}dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^2 dr d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(\theta)d\theta.$$

Now, we have as $\cos(2\theta) = 2\cos^2(\theta)-1$

$$\cos^3(\theta) = \frac{1}{2}\cos(\theta)\left(\cos(2\theta)+1\right) = \frac{1}{4}\left(\cos(3\theta)+\cos(\theta)\right)+\frac{1}{2}\cos(\theta) = \frac{1}{4}\cos(3\theta)+\frac{3}{4}\cos(\theta).$$

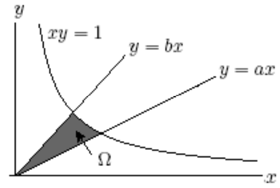
Furthermore, as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)d\theta = [\sin(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(3\theta)d\theta = \left[\frac{\sin(3\theta)}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{2}{3},$$

we obtain

$$\int_D \sqrt{x^2+y^2}dxdy = \frac{8}{3}\left(-\frac{1}{4}\times\frac{2}{3}+\frac{3}{4}\times 2\right) = \frac{32}{9}$$

13.2. Fubini's theorem. Let $0 < a < b < \infty$ and Ω the domain defined as below :



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With formulas, this means that

$$\Omega = \mathbb{R}^2 \cap \{(x,y) : x \geq 0,\ y \geq 0,\ 0 \leq xy \leq 1,\ ax \leq y \leq bx\}$$

Show that

$$\int_\Omega xy\,dxdy = \frac{1}{4}\log\left(\frac{b}{a}\right).$$

Solution: The extremities are

$$xy = 1 \text{ und } y = bx \to x = \frac{1}{\sqrt{b}}$$

$$xy = 1 \text{ und } y = ax \to x = \frac{1}{\sqrt{a}}.$$

We split that domain in two parts $0 \le x \le \frac{1}{\sqrt{b}}$ and $\frac{1}{\sqrt{b}} \le x \le \frac{1}{\sqrt{a}}$ and we get

$$\int_\Omega xy\,dxdy = \int_0^{\frac{1}{\sqrt{b}}}\int_{ax}^{bx} xy\,dydx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\int_{ax}^{\frac{1}{x}} xy\,dydx$$

By Fubini's theorem, one obtains

$$\begin{aligned}
\int_\Omega xy\,dxdy &= \int_0^{\frac{1}{\sqrt{b}}}\int_{ax}^{bx} xy\,dydx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\int_{ax}^{\frac{1}{x}} xy\,dydx \\
&= \int_0^{\frac{1}{\sqrt{b}}} x\left[\frac{y^2}{2}\right]_{ax}^{bx} dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} x\left[\frac{y^2}{2}\right]_{ax}^{\frac{1}{x}} dx \\
&= \frac{1}{2}\int_0^{\frac{1}{\sqrt{b}}}\left(x(bx)^2 - x(ax)^2\right)dx + \frac{1}{2}\int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\left(x\left(\frac{1}{x}\right)^2 - x(ax)^2\right)dx \\
&= \frac{b^2}{2}\int_0^{\frac{1}{\sqrt{b}}} x^3dx + \frac{1}{2}\int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\frac{1}{x}dx - \frac{a^2}{2}\int_0^{\frac{1}{\sqrt{a}}} x^3dx \\
&= \frac{b^2}{2}\left[\frac{x^4}{4}\right]_0^{\frac{1}{\sqrt{b}}} + \frac{1}{2}\Big[\ln|x|\Big]_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} - \frac{a^2}{2}\left[\frac{x^4}{4}\right]_0^{\frac{1}{\sqrt{a}}} \\
&= \frac{b^2}{2}\frac{1}{4b^2} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{a}}\right) - \frac{1}{2}\ln\left(\frac{1}{\sqrt{b}}\right) - \frac{a^2}{2}\frac{1}{4a^2} \\
&= \frac{1}{8} + \frac{1}{2}\log\left(\frac{1}{\sqrt{a}}\right) - \frac{1}{2}\log\left(\frac{1}{\sqrt{b}}\right) - \frac{1}{8} \\
&= \frac{1}{2}\left(\log\left(\frac{1}{\sqrt{a}}\right) - \log\left(\frac{1}{\sqrt{b}}\right)\right) = \frac{1}{4}\log\left(\frac{b}{a}\right).
\end{aligned}$$

13.3. Centre of mass. Compute the centre of mass of a half ball of radius $R > 0$

$$B^3_+(0,R) = \mathbb{R}^3 \cap \{(x,y,z) : x^2+y^2+z^2 \le R^2, z \ge 0\},$$

defined by

$$\begin{aligned}
\Big(&\frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B_3^+(0,R)} x\,dxdydz, \\
&\frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B_3^+(0,R)} y\,dxdydz, \\
&\frac{1}{\mathrm{Vol}(B^3_+(0,R))}\int_{B_3^+(0,R)} z\,dxdydz\Big),
\end{aligned}$$

where $\mathrm{vol}(B_3^+(0,R))$ is the volume of the half-ball (of radius $R>0$) in 3-space (one can refer to Serie 11 for its value).

Solution: First, by obvious symmetry, we have

$$\frac{1}{\mathrm{Vol}(B_+^3(0,R))}\int_{B_3^+} x\,dxdydz = \frac{1}{\mathrm{Vol}(B_+^3(0,R))}\int_{B_3^+} y\,dxdydz = 0,$$

and as the ball of radius has volume $\frac{4\pi}{3}R^3$, the volume of the half-ball is $\frac{2\pi}{3}R^3$, so we just need to compute the following integral

$$\int_{B_3^+(0,R)} z\,dxdydz.$$

Using spherical coordinates

$$\begin{cases} x = r\cos(\theta)\sin(\varphi) \\ y = r\sin(\theta)\sin(\varphi) \\ z = r\cos(\varphi) \end{cases}$$

where $0 \le r \le R$, $-\pi \le \theta \le \pi$ and $0 \le \varphi \le \frac{\pi}{2}$ (notice that the angle φ must be positive). Now, we have (using $2\cos(\theta)\sin(\theta) = 2\sin(2\theta)$)

$$\begin{aligned}
\int_{B_3^+(0,R)} z\,dxdydz &= \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}}\int_0^R r^3\cos(\varphi)\sin(\varphi)drd\varphi d\theta \\
&= 2\pi\left(\int_0^R r^3 dr\right)\left(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\varphi)d\varphi\right) \\
&= 2\pi\left[\frac{r^4}{r}\right]_0^R\left[-\frac{1}{4}\cos(2\varphi)\right]_0^{\frac{\pi}{2}} \\
&= 2\pi \times \frac{R^4}{4} \times \frac{1}{2} \\
&= \frac{\pi R^4}{4}.
\end{aligned}$$

As the volume as the upper half-ball of radius $R>0$ in $\mathbb{R}^3$ is $\frac{2\pi}{3}R^3$, we find

$$\frac{1}{\mathrm{vol}(B_3^+(0,R))}\int_{B_3^+(0,R)} z\,dxdydz = \frac{3}{2\pi R^3}\times\frac{\pi R^4}{4} = \frac{3R}{8} < R$$

as it should be. Therefore, the centre of gravity (or mass) of the upper-half ball of radius $R>0$ in $\mathbb{R}^3$ is

$$\left(0, 0, \frac{3R}{8}\right).$$

13.4. The Astroid Let $a>0$. The Astroid $A(a)\subset\mathbb{R}^2$ is the geometric figure in the plane defined by

$$A(a) := \left\{(x,y)\in\mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right\}$$

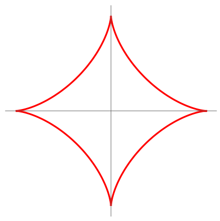

Abbildung 1: The red Astroid $A(a)$ is the boundary of $B(a)$.

The construction of an Astroid is very geometric (see link). Let $B(a)$ denote the set

$$B(a) = \{(rx, ry) \subset \mathbb{R}^2 \mid r \in [0,1],\ (x,y) \in A(a)\}.$$

Compute the area of $B(a)$ using the theorem of Green. Note that the theorem of Green is indeed applicable in this case.

Solution: A parametrisation of $A(a)$ is such that $\gamma : [0, 2\pi] \to \mathbb{R}^2$,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (a\cos^3(t), a\sin^3(t)).$$

Thanks of Green's theorem, we have

$$\begin{aligned}\text{Area}(B(a)) &= \iint_{B(a)} 1\ dxdy = \int_{A(a)} (0, x) \cdot d\vec{s} \\ &= \int_0^{2\pi} \gamma_1(t)\dot{\gamma}_2(t)\ dt \\ &= 3a^2 \int_0^{2\pi} \cos^4(t)\sin^2(t)\ dt\end{aligned}$$

Now, compute that

$$\cos^4(\theta)\sin^2(\theta) = \frac{1}{32}\left(2 + \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)\right)$$

and as for all integer $k \geq 1$, we have

$$\int_0^{2\pi} \cos(k\theta)d\theta = 0,$$

we deduce that

$$\int_0^{2\pi} \cos^4(t)\sin^2(t)\ dt = \frac{\pi}{8},$$

so that

$$\text{Area}(B(a)) = \frac{3\pi a^2}{8}.$$