14.1. Multiple choice questions

(1) There is a unique solution f of the differential equation

$$y'' + (x^2+1)y' + y = 0$$

such that $f(-1) = -1$.

True ☐ False ☑

Solution. This is false, as there is a 1-dimensional vector space of solutions by Cauchy-Lipschitz.

(2) If f is C^2 on $\mathbb{R}^2$ and f is maximal at $(0,0)$, then

$$\partial_x f(0,0) + \partial_y f(0,0) = 0.$$

True ☑ False ☐

Solution. Since $\mathcal{D}f(0,0) = 0$, the claim follows trivially by taking the sum of partial derivatives.

(3) Let f be a C^2 function $\mathbb{R}^2 \to \mathbb{R}$ and define $g(u,v) = f(u+v, u-v)$. We have

$$\partial_{v^2} g = \partial_{x^2} f(u+v, u-v) + \partial_{y^2} f(u+v, u-v).$$

True ☐ False ☑

Solution. This is indeed false, since

$$\begin{aligned}
\partial_v g(u,v) &= \partial_x f(u+v, u-v) - \partial_y f(u+v, u-v) \\
\partial_{v^2} g(u,v) &= \partial_{x^2} f(u+v, u-v) - 2\,\partial_{xy} f(u+v, u-v) + \partial_{y^2} f(u+v, u-v).
\end{aligned}$$

(4) If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^2, $\nabla f(0,0,0) = 0$, and the Hessian matrix of f at $(0,0,0)$ is

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Then f has at $(0,0,0)$

A local minimum ☑
A local maximum ☐
A saddle point ☐

Solution. We have $4 > 0$,

$$\det \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} = 15 > 0 \qquad \det \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{pmatrix} = 41 > 0.$$

Therefore, by the Sylvester criterion, f admits a local minimum at $(0,0,0)$.

(5) The center of mass of a compact subset $X \subset \mathbb{R}^2$ of positive area ($\text{Area}(X) > 0$) is equal to

$$(x_0, y_0) = \left(\frac{1}{\text{Area}(X)} \int_X x \, dxdy, \frac{1}{\text{Area}(X)} \int_X y \, dxdy \right).$$

True ☑ False ☐

Solution. Indeed, this is just the definition.

(6) Let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field. The vector field is conservative if and only if $\text{curl}(f) = 0$.

True ☑ False ☐

Solution. This exactly means that the 1-form associated to f is closed, and $\mathbb{R}^3$ is simply connected.

14.2. ODE

Find the solution f of the ODE

$$y'' + y' - 6y = x$$

such that $f(0) = 1$ and $f'(0) = 0$.

Solution. Since

$$X^2 + X - 6 = (X - 2)(X + 3),$$

the solutions of the homogeneous equation are

$$y(x) = \lambda_1 e^{2x} + \lambda_2 e^{-3x}.$$

As the right-hand side is a polynomial, we can look for a particular solution of the form

$$y_0(x) = ax + b$$

for some $a, b \in \mathbb{R}$. Then we get

$$y_0'' + y_0' - 6y_0 = a - 6(ax + b) = -6ax + (a - 6b) = x,$$

so that

$$a = -\frac{1}{6}, \qquad b = -\frac{1}{36}.$$

Therefore, the solutions of the equation are

$$y(x) = \lambda_1 e^{2x} + \lambda_2 e^{-3x} - \frac{1}{36}(6x + 1).$$

Now, we have

$$y(0) = \lambda_1 + \lambda_2 - \frac{1}{36}$$
$$y'(0) = 2\lambda_2 - 3\lambda_2 - \frac{1}{6}.$$

Therefore, y solves the initial conditions if and only if

$$\lambda_1 = \frac{117}{5\cdot 36} = \frac{13}{20} \qquad \text{and} \qquad \lambda_2 = \frac{68}{5\cdot 36} = \frac{17}{45}.$$

Finally, the solution is

$$f(x) = \frac{13}{20}e^{2x} + \frac{17}{45}e^{-3x} - \frac{1}{36}(6x+1).$$

14.3. Hessian

Compute the Hessian of $f(x,y,z) = \sqrt{x^4+y^4+z^2+1}$ at $(x,y,z) = (0,0,0)$.

Solution. As $\sqrt{1+t} = 1 + \frac{t}{2} + O(t^2)$, we find

$$f(x,y,z) = 1 + \frac{1}{2}\left(x^4+y^4+z^2\right) + O((x^4+y^4+z^2)^2) = 1 + \frac{1}{2}z^2 + O((x^2+y^2+z^2)^2).$$

Therefore, we have

$$\operatorname{Hess} f(0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

14.4. Integral

Compute the integral

$$\int_{D^2} x^2y^2\,dxdy$$

where $D^2 \subset \mathbb{R}^2$ is the unit disk defined by

$$D^2 = \mathbb{R}^2 \cap \{(x,y) : x^2+y^2 \le 1\}.$$

Solution. Using polar coordinates, we find

$$\begin{aligned}\int_{D^2} x^2y^2\,dxdy &= \left(\int_0^1 r^5 dr\right)\left(\int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)d\theta\right) = \frac{1}{24}\int_0^{2\pi}\sin^2(2\theta)d\theta \\ &= \frac{1}{48}\int_0^{4\pi}\sin^2(t)dt = \frac{1}{24}\int_0^{2\pi}\sin^2(t)dt = \frac{\pi}{24}.\end{aligned}$$

14.5. Taylor polynomial

Compute the Taylor formula at order 2 of $f(x,y,z) = \cos\left(\frac{x}{1+y^2} - \frac{y}{1+z^2}\right)$ at $(x,y,z) = (0,0)$.

Solution. Recall that

$$\cos(t) = 1 - \frac{t^2}{2} + O(t^4).$$

Therefore, we have

$$\begin{aligned}
f(x,y,z) &= 1 - \frac{1}{2}\left(\frac{x^2}{(1+y^2)^2} + \frac{y^2}{(1+z^2)^2} - \frac{2xy}{(1+y^2)(1+z^2)}\right) + O((x^2+y^2+z^2)^2) \\
&= 1 - \frac{1}{2}\left(x^2 + y^2 - 2xy\right) + O((x^2+y^2+z^2)^2) \\
&= 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + xy + O((x^2+y^2+z^2)^2).
\end{aligned}$$

Therefore, the Taylor polynomial $T^2_{(0,0,0)}f$ of order 2 of f at $(x,y,z) = (0,0,0)$ is given by

$$T^2_{(0,0,0)}f(x,y,z) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + xy.$$

14.6. Critical points

Let $U = \mathbb{R}^2 \cap \{(x,y) \mid x > 0,\ y > 0\}$. Define

$$f(x,y) = \frac{y}{2x} + \frac{x-1}{y^2}$$

for all $(x,y) \in U$. Find the values of $(x,y) \in \mathbb{R}^2$ such that f has a critical point at (x,y). Determine whether they are a local maximum, a local minimum or a saddle point.

Solution.

We have

$$\mathcal{D}f(x,y) = \left(-\frac{y}{2x^2} + \frac{1}{y^2}, \frac{1}{2x} - \frac{2(x-1)}{y^3}\right) = (0,0)$$

if and only if

$$\begin{cases} -\frac{1}{2}y^3 + x^2 = 0 \\ \frac{1}{2}y^3 - 2x(x-1) = 0 \end{cases}$$

which yields (as $x > 0$ and y is real)

$$(x,y) = (2,2).$$

Now, we have

$$\mathcal{D}^2 f(x,y) = \begin{pmatrix} \frac{y}{x^3} & -\frac{1}{2x^2} - \frac{2}{y^3} \\ -\frac{1}{2x^2} - \frac{2}{y^3} & \frac{6(x-1)}{y^4} \end{pmatrix}.$$

Therefore, we have

$$\mathcal{D}^2 f(2,2) = \begin{pmatrix} \frac{1}{4} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

and

$$\det \mathcal{D}^2 f(2,2) = -\frac{3}{64} < 0$$

which implies that $\mathcal{D}^2 f(2,2)$ admits a strictly positive eigenvalue and a strictly negative eigenvalue. We conclude that $(2,2)$ is a saddle point of f.

14.7. Vector field

(1) Check that the vector-field $\mathbb{R}^2$

$$f(x,y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$$

is conservative.

(2) Compute a potential of f.

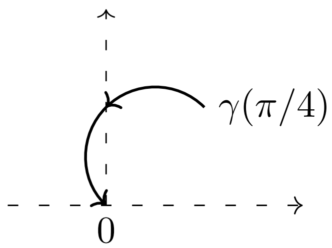
(3) Compute

$$\int_\gamma f \cdot d\vec{s},$$

where γ is the parametrised curve

$$\begin{cases} \gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \mathbb{R}^2 \\ \theta \mapsto \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(\theta), \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\theta)\right). \end{cases}$$

Oriented path of γ



Solution.

(1) One can check directly that

$$\partial_y\left(2xy^2 - 5x^4y + 5\right) = 4xy - 5x^4 = \partial_x\left(-7y^6 - x^5 + 2x^2y\right).$$

As $\mathbb{R}^2$ is simply connected (star-shaped), we deduce that f is conservative.

(2) A potential is given by $\omega(x,y) = x^2y^2 - x^5y + 5x - y^7$.

(3) As f is conservative, we have

$$\begin{aligned}\int_\gamma f \cdot d\vec{s} &= \varphi\left(\gamma\left(\frac{5\pi}{2}\right)\right) - \varphi\left(\gamma\left(\frac{\pi}{4}\right)\right) \\ &= \varphi(0,0) - \varphi(1,1) = -(1 - 1 + 5 - 1) \\ &= -4.\end{aligned}$$

14.8. Line integral

Let $f(x,y) = (\cos(xy), \sin(xy))$. Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

along the rectangle with vertices $(0,0)$, $(0,\pi)$, (π,π) and $(\pi,0)$ oriented counterclockwise.

Solution. We let $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ parametrise each arc above, such that for all $0\le t\le\pi$

$$\begin{cases}\gamma_1(t)=(t,0)\\ \gamma_2(t)=(\pi,t)\\ \gamma_3(t)=(\pi-t,\pi)\\ \gamma_4(t)=(0,\pi-t).\end{cases}$$

Then we have

$$\begin{aligned}\int_{\gamma_1} f(s)\cdot d\vec{s} &= \int_0^\pi \langle(1,0),(1,0)\rangle dt=\pi\\ \int_{\gamma_2} f(s)\cdot d\vec{s} &= \int_0^\pi \langle(\cos(\pi t),\sin(\pi t)),(0,1)\rangle dt=\int_0^\pi \sin(\pi t)dt=\left[-\frac{1}{\pi}\cos(\pi t)\right]_0^\pi\\ &=\frac{1}{\pi}(1-\cos(\pi^2))\\ \int_{\gamma_3} f(s)\cdot d\vec{s} &= \int_0^\pi \langle(\cos(\pi(\pi-t)),\sin(\pi(\pi-t))),(-1,0)\rangle dt=\int_0^\pi -\cos(\pi(\pi-t))dt\\ &=\left[\frac{1}{\pi}\sin(\pi(\pi-t))\right]_0^\pi=-\frac{1}{\pi}\sin(\pi^2)\\ \int_{\gamma_4} f(s)\cdot d\vec{s} &= \int_0^\pi \langle(1,0),(0,-1)\rangle dt=0.\end{aligned}$$

Finally, we have

$$\int_\gamma f(s)\cdot d\vec{s}=\sum_{i=1}^4\int_{\gamma_i} f(s)\cdot d\vec{s}=\pi+\frac{1}{\pi}-\frac{1}{\pi}\big(\cos(\pi^2)+\sin(\pi^2)\big).$$

14.9. Integral

(1) Show that for all $\theta\in\mathbb{R}$, we have

$$\cos(\theta)\sin^3(\theta)=-\frac{1}{8}\sin(4\theta)+\frac{1}{4}\sin(2\theta).$$

(2) Compute the integral

$$\int_{B^3_+} x^2z\,dxdydz,$$

where $B^3_+\subset\mathbb{R}^3$ is the upper half-ball

$$B^3_+=\mathbb{R}^3\cap\{(x,y,z):x^2+y^2+z^2\le 1, z\ge 0\}.$$

Solution.

(1) We have by Euler's formula

$$\begin{aligned}\cos(\theta)\sin^3(\theta)&=\frac{e^{i\theta}+e^{-i\theta}}{2}\frac{e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}}{-8i}\\ &=-\frac{1}{16i}\left(e^{4i\theta}-e^{-4i\theta}-2e^{3i\theta}+2e^{-3i\theta})\right)\\ &=-\frac{1}{8}\sin(4\theta)+\frac{1}{4}\sin(2\theta).\end{aligned}$$

(2) Recall that spherical coordinates are given on $\mathbb{R}^3 \setminus \{0\}$ by

$$\begin{aligned} F : \mathbb{R}_+^* \times [0, 2\pi) \times [0, \pi) &\to \mathbb{R}^3 \\ (r, \theta, \varphi) &\mapsto (r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)). \end{aligned}$$

As we want to parametrise $B_+^3 \cap \{(x, y, z) : z \geq 0\}$, this imposes $0 \leq r \leq 1$ and $r\cos(\varphi) \geq 0$, or equivalently $0 \leq \varphi \leq \frac{\pi}{2}$. Recall, (or compute) that the Jacobian determinant admits the following expression

$$\operatorname{Jac} F(r, \theta, \varphi) = |\det \mathcal{D}^2 F(r, \theta, \varphi)| = r^2 |\sin(\varphi)| = r^2 \sin(\varphi),$$

where we used that $\sin(\varphi) \geq 0$ for $0 \leq \varphi \leq \dfrac{\pi}{2}$. Therefore, the change of formula (neglecting zero measure subsets) and Fubini yield

$$\begin{aligned} \int_{B_+^3} x^2 z \, dxdydz &= \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^2 \cos^2(\theta) \sin^2(\varphi) \times r\cos(\varphi) \times r^2 \sin(\varphi) \, drd\theta d\varphi \\ &= \left(\int_0^1 r^5 dr\right) \left(\int_0^{2\pi} \cos^2(\theta) d\theta\right) \left(\int_0^{\frac{\pi}{2}} \cos(\varphi) \sin^3(\varphi) d\varphi\right). \end{aligned}$$

Now we trivially have

$$\int_0^1 r^5 dr = \frac{1}{6}, \qquad \int_0^{2\pi} \cos^2(\theta) d\theta = \int_0^{2\pi} \left(\frac{1 + \cos(2\theta)}{2}\right) d\theta = \pi,$$

while

$$\int_0^{\frac{\pi}{2}} \cos(\varphi) \sin^3(\varphi) d\varphi = \left[\frac{1}{4} \sin^4(\varphi)\right]_0^{\frac{\pi}{2}} = \frac{1}{4}.$$

Finally, we deduce that

$$\int_{B_+^3} x^2 z \, dxdydz = \frac{1}{6} \times \pi \times \frac{1}{4} = \frac{\pi}{24}.$$

Notice that it checks with (1) as

$$\int_0^{\frac{\pi}{2}} \left(-\frac{1}{8}\cos(4\theta) + \frac{1}{4}\sin(2\theta)\right) d\theta = \left[-\frac{1}{8}\cos(2\theta)\right]_0^{\frac{\pi}{2}} = \frac{1}{4}.$$

14.10. Epicycloids

Let $r > 0$ and $p \in \mathbb{N} \setminus \{0\}$, and let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the closed curve given by the parametric equation

$$\gamma(\theta) = (r\left((p+1)\cos(\theta) - \cos((p+1)\theta)\right), r\left((p+1)\sin(\theta) - \sin\left((p+1)\theta\right)\right)).$$

(1) Show that for all $\theta \in [0, 2\pi]$, we have

$$\cos(\theta)\cos((p+1)\theta) + \sin(\theta)\sin((p+1)\theta) = \cos(p\theta)$$

(2) Compute the length of $\gamma([0, 2\pi]) \subset \mathbb{R}^2$ which is given by

$$\operatorname{Length}(\gamma) = \int_0^{2\pi} |\gamma'(\theta)| d\theta.$$

(3) Compute the area of the compact region delimited by the closed curve $\gamma([0, 2\pi])$.

Solution.

(1) We have for all $a_1, a_2, b_1, b_2 \in \mathbb{C}$

$$(a_1 + a_2)(b_1 + b_2) - (a_1 - a_2)(b_1 - b_2) = 2(a_1 b_2 + a_2 b_1),$$

so the Euler formulas imply with $a_1 = e^{i\theta}$, $a_2 = e^{-i\theta}$, $b_1 = e^{i(p+1)\theta}$, $b_2 = e^{-i(p+1)\theta}$ that

$$\cos(\theta)\cos((p+1)\theta) + \sin(\theta)\sin((p+1)\theta) = \frac{1}{4} \times 2\left(e^{ip\theta} + e^{-ip\theta}\right) = \cos(p\theta).$$

(2) We first compute

$$\begin{aligned}
|\gamma'(\theta)|^2 = r^2\Big(&(p+1)^2\sin^2(\theta) + (p+1)^2\sin^2((p+1)\theta) - 2(p+1)^2\sin(\theta)\sin((p+1)\theta) \\
&+ (p+1)^2\cos^2(\theta) + (p+1)^2\cos^2((p+1)\theta) - 2(p+1)^2\cos(\theta)\cos((p+1)\theta)\Big) \\
&= 2(p+1)^2r^2\left(1 - \cos(\theta)\cos((p+1)\theta) - \sin(\theta)\sin((p+1)\theta)\right) \\
&= 2(p+1)^2r^2\left(1 - \cos(p\theta)\right).
\end{aligned}$$

Therefore, the duplication formula $\cos(2t) = 1 - 2\sin^2(t)$ implies that

$$\begin{aligned}
\mathrm{Length}(\gamma) = \int_0^{2\pi} |\gamma'(\theta)| d\theta &= \sqrt{2}(p+1)r\int_0^{2\pi}\sqrt{1 - \cos(p\theta)}d\theta \\
&= 2(p+1)r\int_0^{2\pi}\left|\sin\left(\frac{p\theta}{2}\right)\right| d\theta \\
&= 8(p+1)r,
\end{aligned}$$

where we used by symmetry

$$\int_0^{2\pi}\left|\sin\left(\frac{p\theta}{2}\right)\right| d\theta = \frac{2}{p}\int_0^{\pi p}|\sin(t)|dt = \frac{2}{p} \times p\int_0^{\pi}\sin(t)dt = 4.$$

(3) Let Ω_p be the given domain. We have by the Green's formula

$$\begin{aligned}
\mathrm{Area}(\Omega_p) &= \int_\gamma (-y, 0) \cdot d\vec{s} \\
&= (p+1)r^2\int_0^{2\pi}\left(-(p+1)\sin(\theta) + \sin((p+1)\theta)\right)\left(-\sin(\theta) + \sin((p+1)\theta)\right)d\theta \\
&= (p+1)r^2\int_0^{2\pi}\Big((p+1)\sin^2(\theta) - (p+2)\sin(\theta)\sin((p+1)\theta) + \sin^2((p+1)\theta)\Big)d\theta \\
&= \pi(p+1)(p+2)r^2
\end{aligned}$$

where we used

$$\sin(\theta)\sin((p+1)\theta) = \frac{1}{2}\left(\cos(p\theta) - \cos((p+2)\theta)\right)$$
$$\int_0^{2\pi}\sin^2((p+1)\theta)d\theta = \frac{1}{p+1}\int_0^{2\pi(p+1)}\sin^2(s)ds = \int_0^{2\pi}\sin^2(s)ds = \pi$$

and trivial symmetry.

Remark: For the cardioid ($p = 1$), we find a length of $16r$ and area $6\pi r^2$, and for the nephroid ($p = 2$), we find a length of $24r$ and area $12\pi r^2$, which checks with known formulas.