<u>Method 2</u> : "variation of constants"

Assume that <u>$k = 2$</u>

$$y'' + a_1 y' + a_0 y = b \qquad (*)$$

Find a basis (f_1, f_2) of the homogeneous ODE

$$y'' + a_1 y' + a_0 y = 0$$

Then look for a <u>particular</u> solution to $(*)$ of the form

$$f = z_1 f_1 + z_2 f_2$$

where z_1 and z_2 are <u>functions</u>, such that moreover

$$\boxed{z_1' f_1 + z_2' f_2 = 0}$$

Then putting these relations in the equation (*), we get the two equations

$$\begin{cases} z_1' f_1' + z_2' f_2' = b \\ z_1' f_1 + z_2' f_2 = 0 \end{cases}$$

(linear equations for $z_1'(x), z_2'(x)$)

The determinant

$$\begin{vmatrix} f_1' & f_2' \\ f_1 & f_2 \end{vmatrix}$$

is everywhere $\neq 0$ (FACT) so one finds a formula for

$$(z_1'(x),\ z_2'(x))$$

in terms of b, f_1, f_2, f_1', f_2'.

Then z_1, z_2 are primitives of these functions, and $f_0 = z_1 f_1 + z_2 f_2$ is a particular solution. The set of <u>all</u> solutions is the set of functions

particular solution

$$f_0 + t_1 f_1 + t_2 f_2$$

solution of homogeneous equ.

complex numbers

If one wants real-valued solutions (and a_0, a_1 are real) then take a basis (f_1, f_2) which are real-valued and t_1, t_2 in $\mathbb{R}$.

2.6 - Other methods

(1) Change of variable

e.g. if y satisfies an ODE then e^y also with another ODE which might be simpler.

(2) Separation of variable

when the ODE becomes $(g(y))' = b$ for some function g

e.g. $2yy' = b \iff (y^2)' = b$

Chapter III

Differential calculus in $\mathbb{R}^n$

("multivariable calculus")

3.1 - Introduction

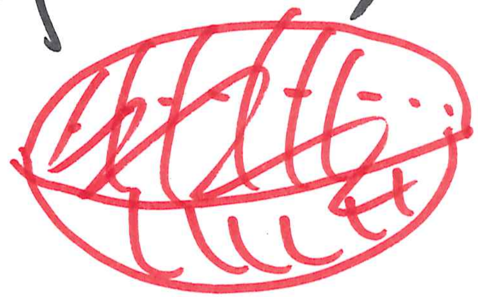
Goal: have tools to study functions of $n \geq 2$ (real) variables

Questions to answer:

(1) how large / how small can a function be?

(2) how does it vary when the variables change?

(3) what is the volume of a solid in $\mathbb{R}^3$ defined by suitable equations?



Examples of functions: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

① **Linear functions** / **affine-linear functions**

$$f(x) = A x + y_0$$

$y_0 \in \mathbb{R}$

$x = (x_1, \ldots, x_n)$

A: matrix
n columns
1 row

more generally:

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f(x) = A x + y_0$$

$y_0 \in \mathbb{R}^m$

A: Matrix
n columns
m rows

② Quadratic functions

$$f(x) = \sum_{1 \le i, j \le n} a_{ij} \, x_i x_j$$

coefficient $\in \mathbb{R}$

$\llcorner (x_1, \dots, x_n)$

ex.

$$f(x_1, x_2) = x_1 x_2$$

$$f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$$

$$= \|x\|^2 = (\text{distance to } 0)^2$$

③ Polynomials

Finite sums of coefficients times monomials in n variables:

$$x_1^{m_1} x_2^{m_2} \cdots\cdots x_n^{m_n}$$

non-negati-ve integers | n variables / coordinates

The degree is $m_1 + \cdots + m_n$; the degree of a polynomial is the largest degree of a monomial.

Ex. $n = 3$

$$f(x) = \underbrace{x_1^3}_{\text{deg } 3} + 12\, \overbrace{x_1^7 x_2 x_3^2}^{\text{degree } 10} - \pi \underbrace{x_1 x_2 x_3}_{\text{degree } 3} + \frac{1}{2} x_2^{10}$$

$\Rightarrow$ $\underline{\deg(f) = 10.}$

Linear functions: degree ≤ 1

Quadratic ——— : degree ≤ 2

4 "<u>Cartesian product" functions</u>:

$f_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_1}$

$f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_2}$

$\rightsquigarrow \quad f = (f_1, f_2) : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_1 + m_2}$

<u>Ex.</u> $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Write $f = (f_1, \ldots, f_m)$, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

(5) "Functions with separated variables"

$$f(x) = f_1(x_1)\, f_2(x_2) \cdots f_n(x_n)$$

$(x_1, \ldots, x_n)$ $\qquad f_i : \mathbb{R} \longrightarrow \mathbb{R}$

Ex.

- monomials (but not polynomials in general)
- $$\exp(-\|x\|^2) = \exp(-x_1^2 - \cdots - x_n^2) = \exp(-x_1^2) \cdots \exp(-x_n^2)$$

density of a gaussian

6. <u>Composition:</u>

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^{m_1} \arrow[r, "g"] & \mathbb{R}^{m_2} \end{tikzcd}$$

~~ex~~ <u>ex</u> : $f_1 + f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f_1 + f_2 = \left(\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{(f_1, f_2)} & \mathbb{R}^2 & \xrightarrow{a} & \mathbb{R} \\ x & \longmapsto & (f_1(x), f_2(x)) & \longmapsto & f_1(x) + f_2(x) \end{array} \right)$$

linear

where $a(y_1, y_2) = y_1 + y_2$.

Visualizations

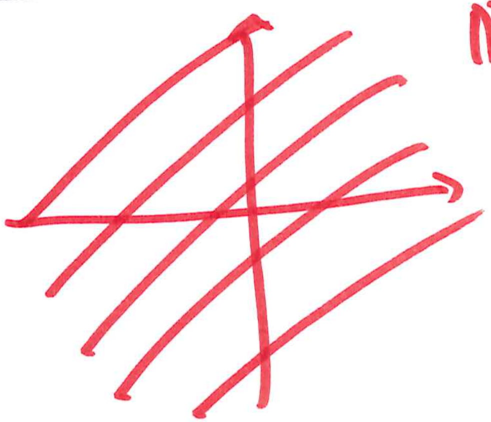
- parametric: $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ or $\mathbb{R}^3$
- graph: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- vector plot: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

3.2 - Continuity / limits

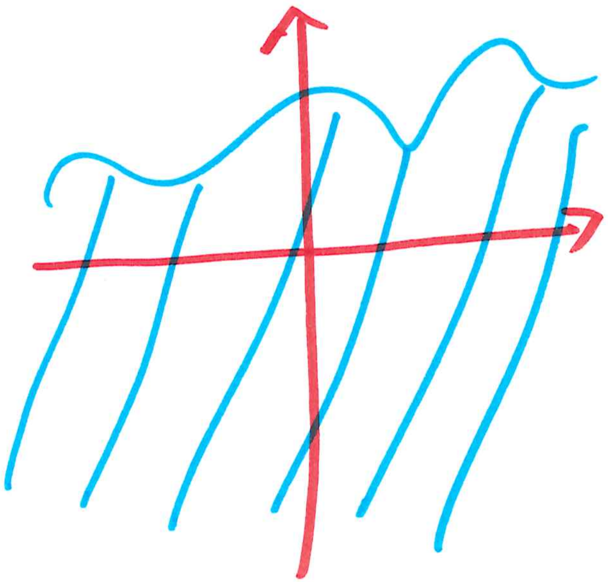
Why is it more complicated with $n \geqslant 2$ variables?

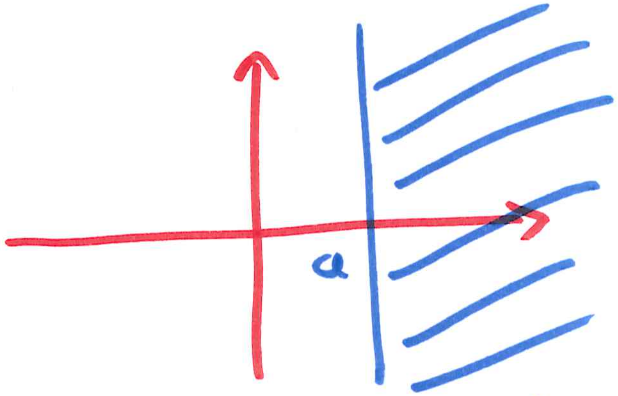
① A function of $n \geqslant 2$ variables can be defined in many different subsets of $\mathbb{R}^n$!

$n = 2$:

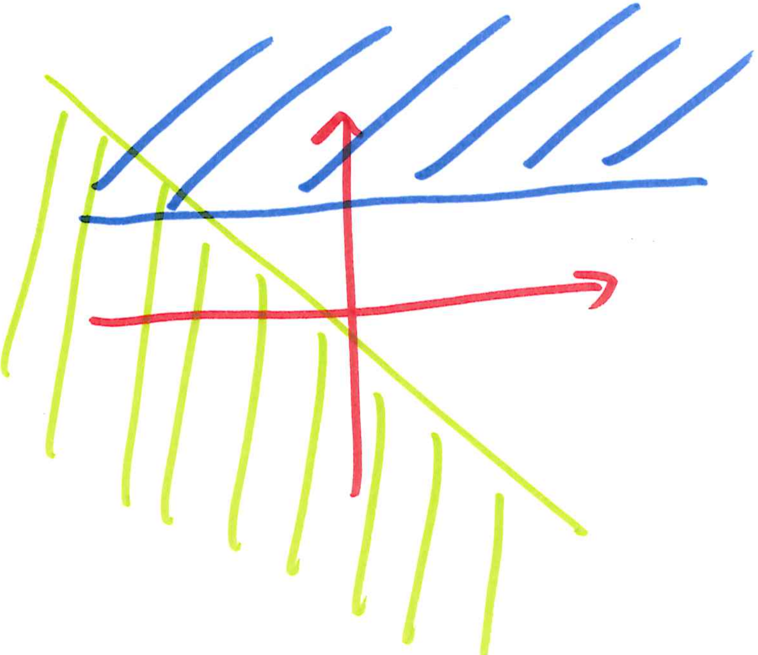


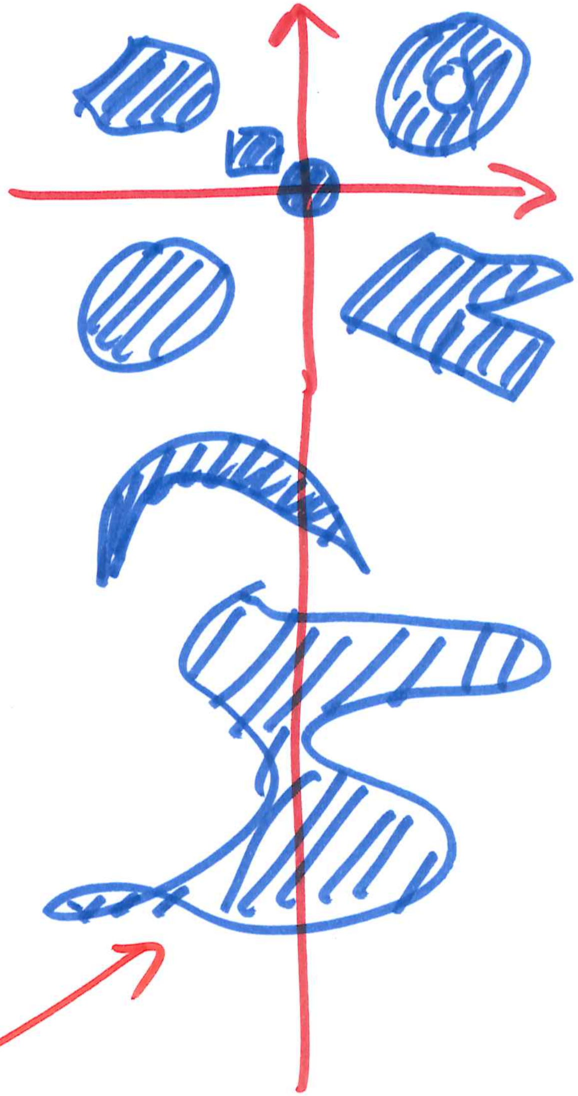
$\mathbb{R}^2$





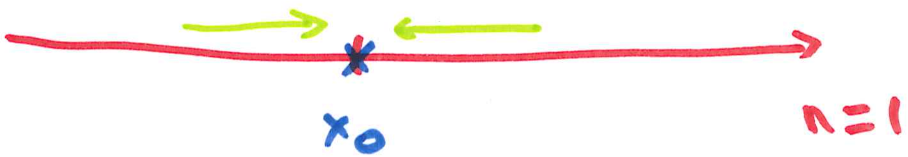
$x_1 \geq a$

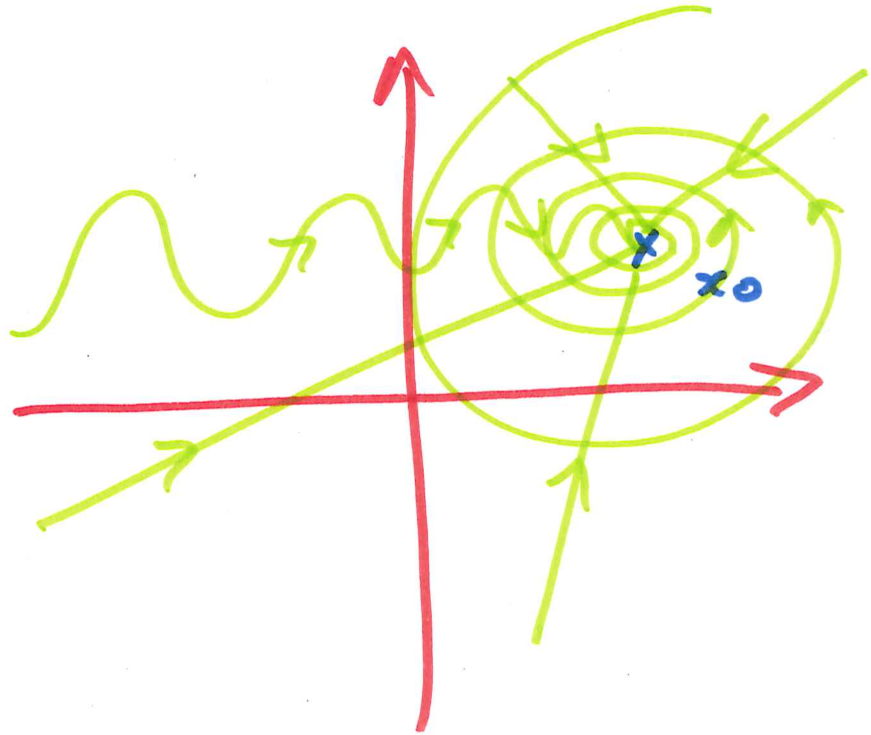




analogues of $[a, b]$

(2) There is more than way to converge to $x_0 \in \mathbb{R}^n$ if $n \geq 2$ than by "straight line".





Definitions (limit of sequences)

$n \geq 1$

$$y_0 = (y_{0,1}, \ldots, y_{0,n})$$

For $k \geq 1$, $x_k = (x_{k,1}, \ldots, x_{k,n})$

$$\boxed{\lim_{k \to +\infty} x_k = y_0} \iff$$

distance

$$\left(\sum_{i=1}^{n} |x_{k,i} - y_{0,i}|^2\right)^{1/2}$$

① $\forall \varepsilon > 0, \exists k_0 > 0$, if $k \geq k_0$, $\|x_k - y_0\| < \varepsilon$

② $\lim_{k \to +\infty} \|x_k - y_0\| = 0$

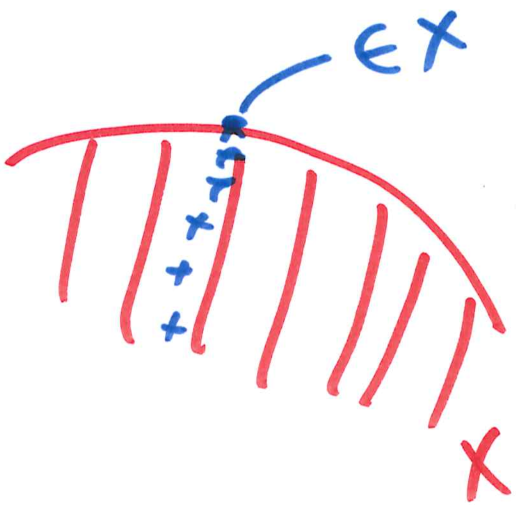
③ For $i = 1, \ldots, n$, $\lim_{k \to +\infty} x_{k,i} = y_{0,i}$

Definition ("compact sets/closed sets")

① A subset $X \subset \mathbb{R}^n$ is <u>closed</u> if for any sequence (x_k) in X, such that $x_k \longrightarrow y_0$ (in $\mathbb{R}^n$), the limit y_0 is in X.

② A subset $X \subset \mathbb{R}^n$ is <u>bounded</u> if $\|x\|$ is bounded as x varies in X.

③ $X \subset \mathbb{R}^n$ is <u>compact</u> if it is <u>closed</u> and <u>bounded</u>.



Ex.

1. $I = [a, b] \subset \mathbb{R}$ is compact

   $]a, b]$ not closed

   $a \leq x_n \leq b$, $x_n \longrightarrow x_0$

   $<$

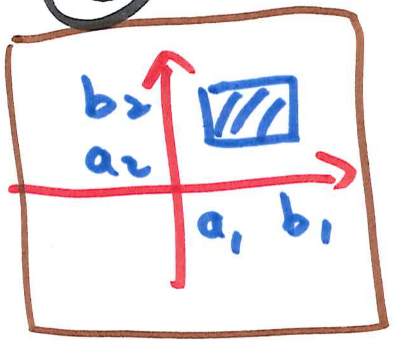
   $\Longrightarrow \quad a \leq x_0 \leq b$

2. $I = [a, +\infty[$ is closed

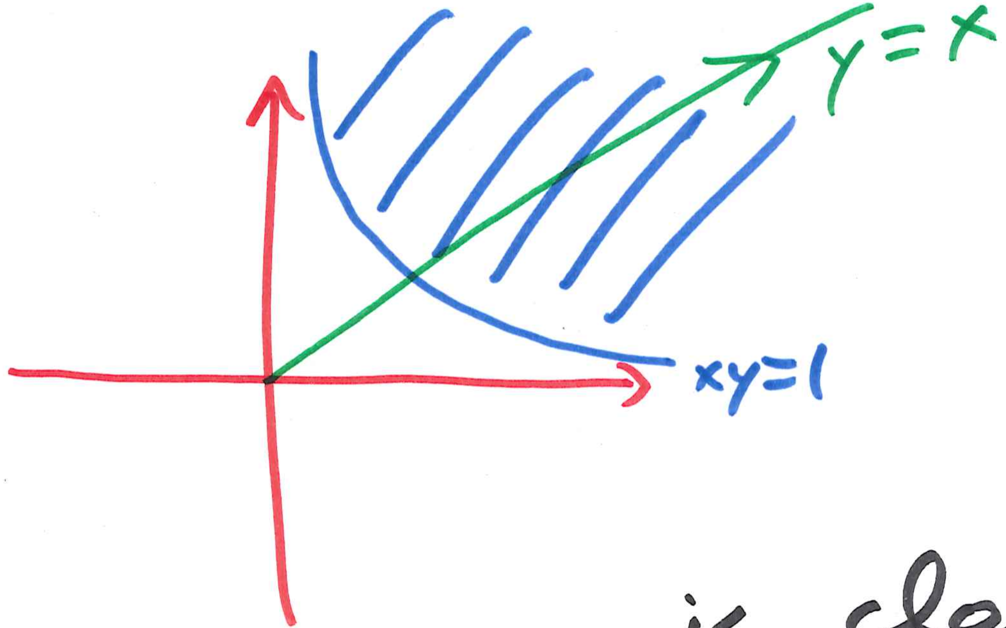
   not compact

3. $n \geq 2$,

   $[a_1, b_1] \times \dots \times [a_n, b_n]$ is closed and bounded (compact)



(4)



(61)

$X = \{(x,y) \in \mathbb{R}^2_+ \mid xy \geq 1\}$

is **closed** but not bounded.

if $(x_k, y_k) \longrightarrow (a,b)$

$\Uparrow$
X

$\rightsquigarrow$ $x_k y_k \geq 1$

(Analysis I) $\Longrightarrow$ $ab \geq 1$

so $(a,b) \in X$

Continuous functions

Def. $X \subset \mathbb{R}^n$

$f : X \longrightarrow \mathbb{R}^m$

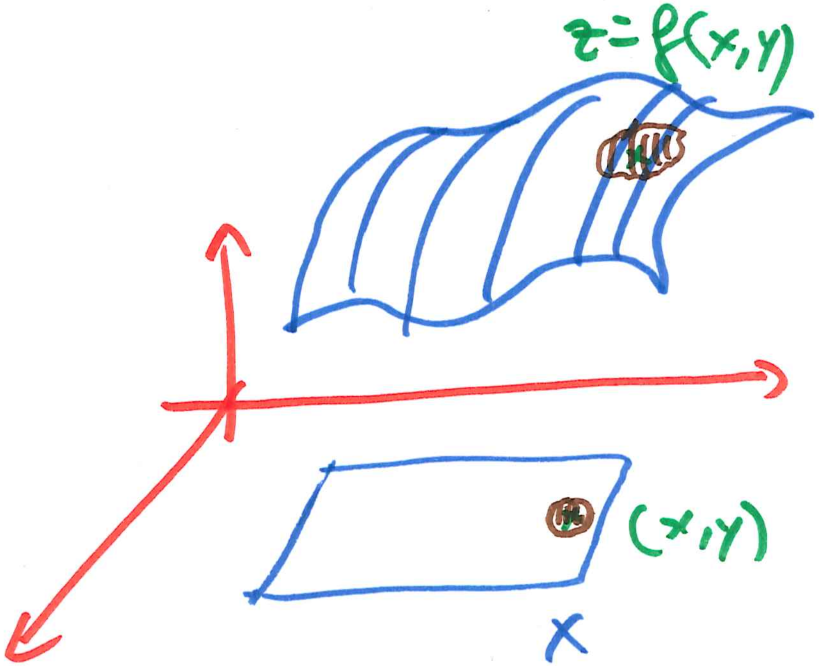
$f = (f_1, \dots, f_m)$

We say that f is <u>continuous</u> on X

$\iff$ each f_i is continuous

$\iff \forall i \, \forall x \in X, \ \forall \varepsilon > 0, \ \exists \delta > 0,$ if $y \in X$ is s.t. $\|x - y\| < \delta$ then $|f_i(x) - f_i(y)| < \varepsilon$

$\iff$ if (x_k) is a sequence in X converging to $x \in X$, then $f(x_k)$ converges to $f(x)$.



(**Note:** $\{ y \in \mathbb{R}^2 \mid \|y - x\| < \delta \}$

= (open) disc centered at x with radius $\delta > 0$

)

<u>Principle:</u> any concrete function obtained by usual rules / composition / usual functions is going to be continuous.
(Careful with denominators!)