Property: if $Y \subset \mathbb{R}^n$ is closed, bounded, **negligible**

then $\boxed{\int_Y f(x_1, \dots, x_n)\, dx_1 \dots dx_n = 0}$

for all $\underline{f : Y \longrightarrow \mathbb{R} \text{ continuous.}}$

Cor. If $X = Y_1 \cup Y_2$, $Y_1 \cap Y_2$ negligible

$$\Rightarrow \int_X f\,dx = \int_{Y_1} f\,dx + \int_{Y_2} f\,dx$$

Examples

(1) [Functions of separated variables / product set]

$$f(x) = f_1(x_1)\, f_2(x_2) \cdots f_n(x_n)$$

$$X = X_1 \times \cdots \times X_n\,, \quad X_i \subset \mathbb{R}$$

$$\Rightarrow \quad \int_X f(x)\,dx = \left(\int_{X_1} f_1(x_1)\,dx_1\right) \cdots \left(\int_{X_2} f_n(x_n)\,dx_n\right)$$

(Ex. $\int_{[0,2]\times[-1,1]} \cos(x)\sin(y)\,dx\,dy$

$= \left(\int_0^2 \cos(x)\,dx\right)\left(\int_{-1}^{1} \sin(y)\,dy\right)$]

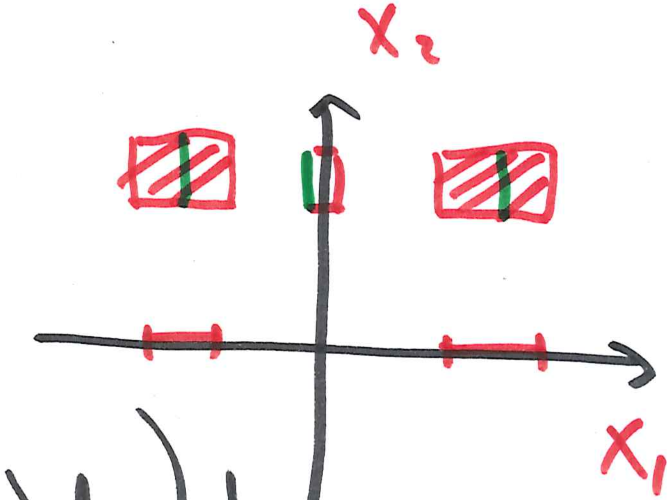
$\underline{n=2}$: $X = X_1 \times X_2$, $2 = 1+1$

X_{x_1} not empty

for $x_1 \in X_1$, and

for all x_1, $X_{x_1} = X_2 \subset \mathbb{R}$

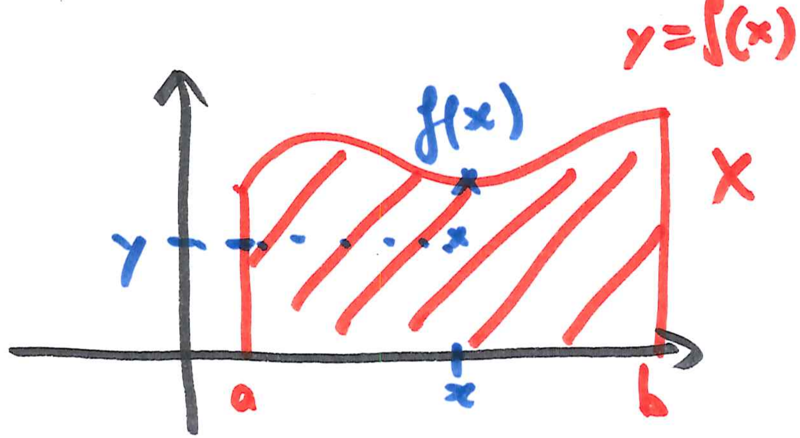
$\overset{\Longrightarrow}{\text{Fubini}} \int f\,dx_1\,dx_2 = \int_{X_1}\left(\int_{X_2} f_1(x_1)\,f_2(x_2)\,dx_2\right)dx_1$



$$= \int_{X_1} f_1(x_1) \underbrace{\left(\int_{X_2} f_2\right)}_{\text{number}} dx_1$$

$$= \left(\int_{X_2} f_2\right)\left(\int_{X_1} f_1\right)$$

(2) <u>One - variable:</u>



$$\text{Area}(X) = \int_a^b f(t)\,dt$$

$$\left[\text{here} \quad X = \{(x,y) \mid 0 \le y \le f(x)\} \quad \text{with } f \ge 0\right]$$

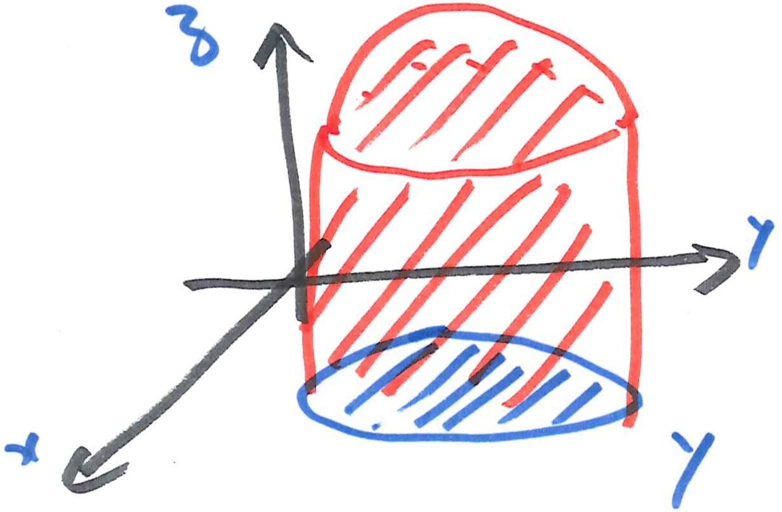
n-variables:

$$Y \subset \mathbb{R}^n, \quad f : Y \longrightarrow \mathbb{R}$$

(closed, bounded) $\qquad f \geq 0$, continuous

$$X = \left\{ (x, y) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} x \in Y \\ 0 \leq y \leq f(x) \end{array} \right\}$$

$\subset \mathbb{R}^{n+1}$, closed, bounded



$$\underset{\substack{\uparrow \\ n+1 \\ \text{dimensional}}}{\mathrm{Vol}}(X) = \int_Y f(x)\,dx$$

Proof: $n+1 = \underbrace{n}_{= n_1} + \underbrace{1}_{= n_2}$

Fubini:

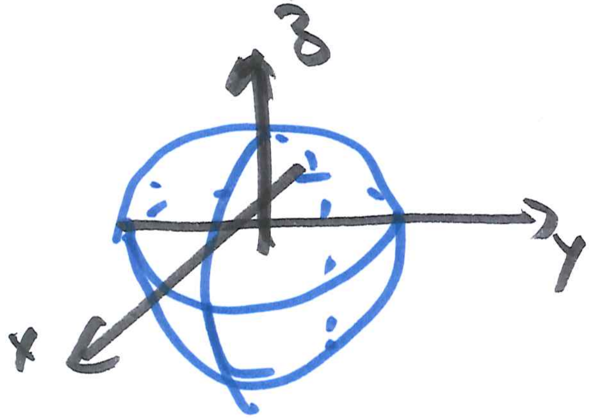
$$\mathrm{Vol}_{(n+1)}(X) = \int_X dx = \int_Y \left(\int_0^{f(x)} 1 \cdot dy \right) dx_1 \dots dx_n$$

$$\left[(x,y) \in X \iff 0 \leq y \leq f(x) \right] = \int_Y f(x)\, dx$$

Ex. Volume of the sphere of radius 1 centered at 0 in $\mathbb{R}^3$:

$$X = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1 \}$$

Method 1: use previous result on $(n+1)$-dim. volumes



$$X = Y_1 \cup Y_2$$

$$Y_1 = \{ (x,y,z) \mid \begin{array}{l} z \geq 0 \\ x^2+y^2+z^2 \leq 1 \end{array} \}$$

$$Y_2 = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} z \leq 0 \\ x^2+y^2+z^2 \leq 1 \end{array} \right\}$$

$Y_1 \cap Y_2 \subset (x-y)$-plane

so $Y_1 \cap Y_2$ is negligible

$$\Rightarrow \mathrm{Vol}(X) = \mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2)$$

$$= 2\,\mathrm{Vol}(Y_1)$$

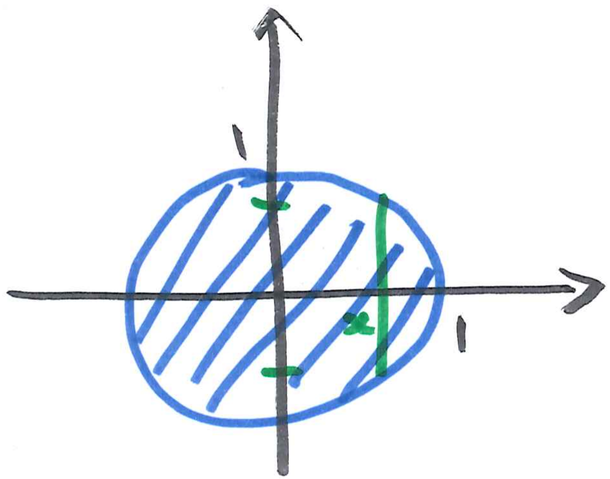
geometrically obvious

$$Y_1 = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,y) \in \text{disc of radius } 1 \\ 0 \leq z \leq \sqrt{1-x^2-y^2} \end{array} \right\}$$

is of the type previously described

so

$$\mathrm{Vol}(Y_1) \overset{(p^o 243)}{=} \int_{(\text{disc})} \sqrt{1-x^2-y^2}\,dx\,dy$$



$$\overset{\text{Fubini}}{=} \int_{-1}^{1} \underbrace{\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy \right)}_{g(x)} dx$$

integral for an area ~~[illegible]~~ using Riemann integral

$$= \frac{1}{2}\left(\text{area of a disc of radius } \sqrt{1-x^2}\right)$$

$$= \frac{\pi}{2}(1-x^2)$$

$$\Rightarrow \mathrm{Vol}(Y_1) = \frac{\pi}{2}\int_{-1}^{1}(1-x^2)\,dx$$

$$= \frac{2\pi}{3}$$

so the volume of the sphere is $\boxed{\frac{4\pi}{3}}$.

<u>Method 2:</u> use Fubini by writing

$$3 = 1 + 2$$

(cutting by a plane x = fixed gives a disc)

$$\mathrm{Vol}(\text{sphere}) = \int_{-1}^{1}\left(\begin{array}{c}\text{area of}\\ \text{disc}\\ \text{of radius } \sqrt{1-x^2}\end{array}\right)dx$$

$$\mathrm{Vol}(\text{sphere}) = \int_{-1}^{1} \pi(1 - x^2)\,dx$$

$$= \frac{4\pi}{3}$$

4.3 _ Improper integrals

(Generalization of $\displaystyle\int_a^{+\infty} f(t)\,dt$

or $\displaystyle\int_{-\infty}^{+\infty} f(t)\,dt$.)

Examples: (X not bounded

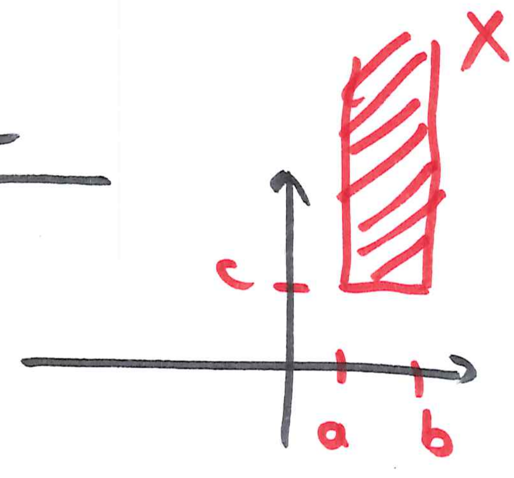
but X is the union

of X_n bounded/closed

and $\int_X f$ "=" $\lim_{n\to\infty} \int_{X_n} f$)

We always assume here that $f \geq 0$

① $X = [a,b] \times [c, +\infty[$



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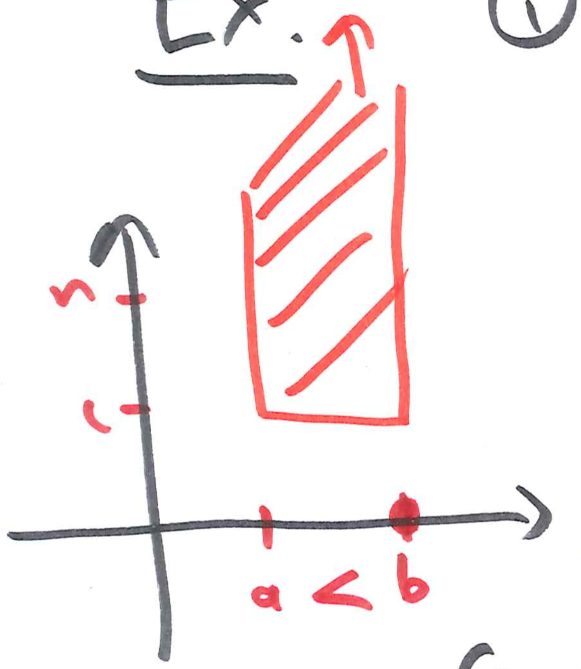
Define: $$\int_X \underbrace{f(x,y)}_{\substack{\text{continuous} \\ (\geq 0)}} \, dx\, dy = \lim_{n\to\infty} \int_{[a,b]\times[c,n]} f(x,y)\, dx\, dy$$

if it exists.

(This is called an "improper" integral)

Ex.



① $\int_X dx\, dy$ does not exist

$\int_{[a,b]\times[c,n]} dx\, dy = (b-a)(n-c)$ has no limit

② $\int_X \frac{1}{y^2}\, dx\, dy = \lim_{n\to\infty} \int_{[a,b]\times[c,n]} \frac{1}{y^2}\, dx\, dy$

$(c>0)$

$$= \lim_{n\to\infty} \left(\int_a^b dx\right)\left(\int_c^n \frac{1}{y^2}\,dy\right)$$

$$= \lim_{n\to\infty} (b-a)\left(\frac{1}{c} - \frac{1}{n}\right) = \frac{b-a}{c}$$

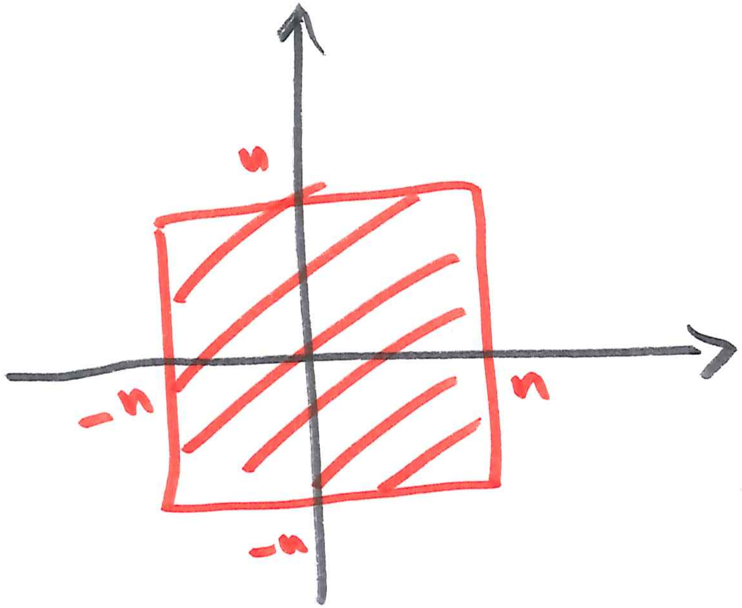
exists

2 $\underline{X = \mathbb{R}^2}$

If $\boxed{f \geq 0}$ and continuous then

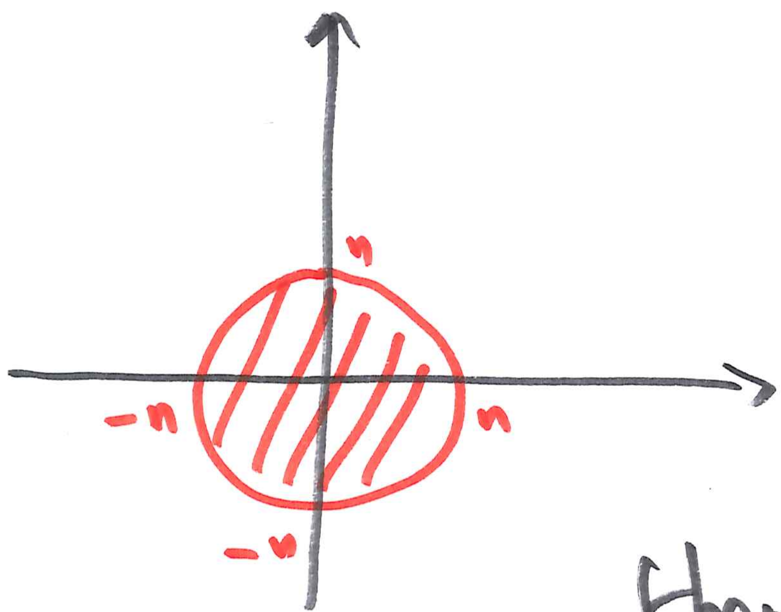
$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy$$

is defined to be the limit, if it exists, of either:



$$\lim_{n\to\infty} \int_{[-n,n]\times[-n,n]} f(x,y)\,dx\,dy$$

or



$$\lim_{n\to\infty} \int_{\begin{pmatrix}\text{disc of}\\ \text{radius}\\ n\end{pmatrix}} f(x,y)\,dx\,dy.$$

[It is true that if one of these limits exist, then also the other, and they are equal.]

Ex. $\displaystyle\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \pi.$

4.4. Change of variable

Generalization of

$$\boxed{\int f(y)\,dy = \int f(\varphi(x))\,\varphi'(x)\,dx}$$

$$y = \varphi(x)$$
$$dy = \varphi'(x)\,dx$$

Situation (in $\mathbb{R}^n$)

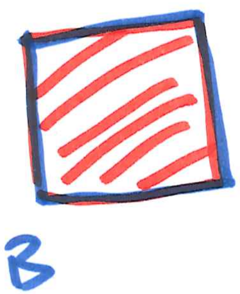
$$X \xrightarrow{\varphi} Y$$

X: bounded, closed — "new coord."

Y: bounded, closed — "old coordinates"

Assume:

$X = \underbrace{X_0}_{\text{open}} \cup \underbrace{B}_{\text{negligible}}$



$Y = \underbrace{Y_0}_{\text{open}} \cup \underbrace{C}_{\text{negligible}}$

and $X_0 \xrightarrow{\varphi} Y_0$

is φ is C^1, bijective

<u>Change of variable formula</u>

$$\boxed{\int_Y f(y)\,dy = \int_X f(\varphi(x))\,|\det J_\varphi(x)|\,dx}$$

$$\begin{cases} y = \varphi(x) \\ dy = |\det J_\varphi(x)|\,dx \end{cases}$$

(<u>if</u> there is a continuous function on X equal to $|\det J_\varphi(x)|$ if $x \in X_0$)

Examples

(1) φ is a translation:

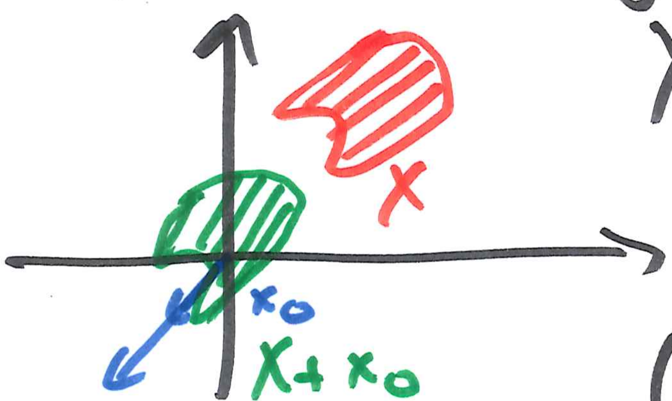
$$\varphi(x) = x + x_0 \quad \text{for some } x_0 \in \mathbb{R}^n$$

and $J_\varphi(x) = \mathrm{Id}$ for all x

so

$$\int_Y f(y)\,dy = \int_X f(x + x_0)\,dx$$

if "$X + x_0 = Y$"
$\varphi(X) = Y$



(in particular: same volume for X and Y)

so for instance the volume of a sphere of radius 1 centered at (x_0, y_0, z_0) is always $\frac{4\pi}{3}$.

(2) $\varphi(x) = Ax$ where A is an invertible matrix

[ex. $\varphi(x_1, \dots, x_n) = (2x_1, \dots, 2x_n)$]

φ is C^1 and

$J_\varphi(x) = A$ for all x

so $|\det J_\varphi(x)| = |\det A|$.

so for $Y \subset \mathbb{R}^n$, $\quad X = A^{-1}Y$

$$\int_Y f(y)\,dy = \int_{A^{-1}Y} f(Ax)\,|\det(A)|\,dx$$

$$= |\det(A)| \int_{A^{-1}Y} f(Ax)\,dx$$

<u>Ex:</u> $f = 1$

$$\mathrm{Vol}(Y) = |\det(A)|\,\mathrm{Vol}(A^{-1}Y)$$

or (putting $X = A^{-1}Y$, $Y = AX$)

$$\mathrm{Vol}(AX) = |\det(A)|\,\mathrm{Vol}(X)$$

<u>For instance:</u>

(1) $X = [0,1]^n \longrightarrow \mathrm{Vol}(X) = 1$

so $\boxed{\mathrm{Vol}(A[0,1]^n) = |\det(A)|}$

(2) $A = \begin{pmatrix} r & & (0) \\ & \ddots & \\ (0) & & r \end{pmatrix}$ $r > 0$
(dilation, $r > 1$)

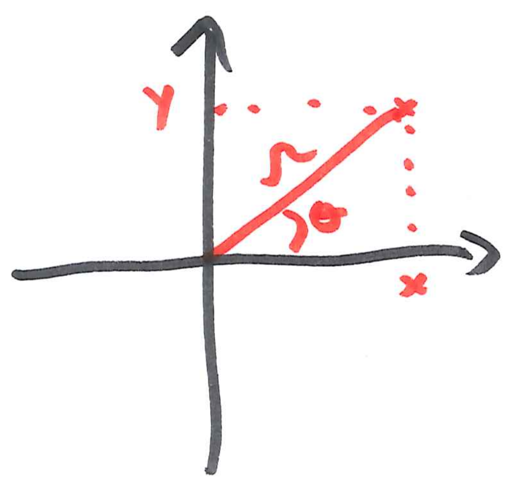
Then $\boxed{\mathrm{Vol}(rX) = r^n \, \mathrm{Vol}(X)}$

(so Vol(sphere of radius $r > 0$) is $\dfrac{4\pi r^3}{3}$)

(3) Polar coordinates

$$[0,+\infty[\times [0,2\pi] \xrightarrow{\varphi} \mathbb{R}^2$$

$$(r, \quad \theta) \qquad\qquad (x,y)$$



$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$\tilde{X}_0 =]0,+\infty[\times]0,2\pi[$ open

φ is C^1 on $\tilde{X}_0$ and

$$J_\varphi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and $\det J_\varphi(r,\theta) = r(\cos^2\theta + \sin^2\theta)$
$= r > 0$

Now to use polar coordinates, to compute:

$$\int_Y f(x,y)\,dx\,dy = \int_X f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where X is chosen so that we can decompose

$$\begin{cases} X = X_0 \cup B \\ Y = Y_0 \cup C \end{cases}$$

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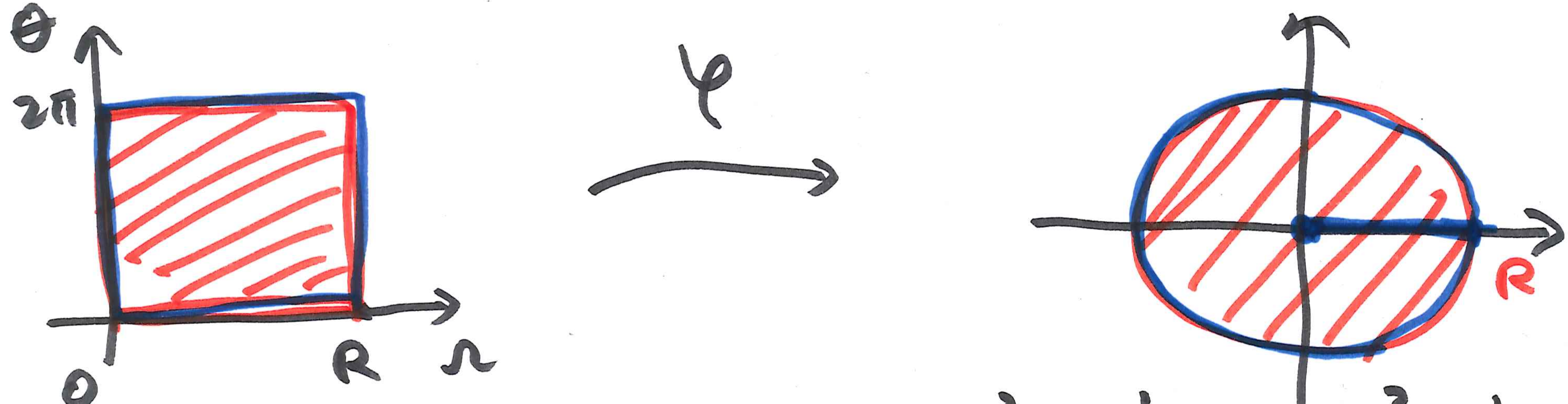
Ex. $Y =$ disc of radius $R > 0$ centered at 0

then $X = [0, R] \times [0, 2\pi]$ works

Then $X_0 =]0, R[\times]0, 2\pi[$



$$Y_0 = \{ (x, y) \mid x^2 + y^2 < R^2,\ y \neq 0 \text{ if } x > 0 \}$$