Def. $f \in C^m(U;\mathbb{R})$, $U \subset \mathbb{R}^n$ open

$k \leq m$, $x_0 \in U$

$$T_k f(y;x_0) = f(x_0) \qquad \text{deg } 0$$

$$+ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)\, y_i \qquad \text{deg } 1$$

$y = (y_1, \dots, y_n)$

$$+ \cdots$$

$$+ \sum_{\substack{m_1+\dots+m_n=j \\ m_i \geq 0}} \frac{1}{m_1! \cdots m_n!} \frac{\partial^j f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) \times y_1^{m_1} \cdots y_n^{m_n} \qquad \text{deg } j$$

multiplication

(multi-indices)

$$+ \cdots \quad (\text{up to } j = k)$$

Ex. For $n=1$, we recover the usual Taylor polynomials

Notation.

$$\begin{cases} m = (m_1, \dots, m_n) & \text{multi-index} \\ |m| = m_1 + \dots + m_n & \\ m! = m_1! \cdots m_n! & \\ y^m = y_1^{m_1} \cdots y_n^{m_n} & \end{cases}$$

so

$$T_k f(y; x_0) = f(x_0) + \sum_{j=1}^{k} \sum_{|m| = j} \frac{1}{m!} \partial^{j}_{x^m} f(x_0)\, y^m$$

$\partial_{x_1^{m_1}} \cdots \partial_{x_n^{m_n}} f$

all multi-indices with $|m| = j$

(15)

Ex. (1) $\underline{n=2}$ $\quad x, y$ variables in $\mathbb{R}^2$

u, v ——————— of polynomial

Term of degree j at (x_0, y_0):

$$\sum_{a=0}^{j} \frac{1}{\underset{m_1!}{a!}\,\underset{m_2!}{(j-a)!}} \frac{\partial^j f}{\partial_x^a \partial_y^{j-a}}(x_0, y_0)\, u^a v^{j-a}$$

(2) arbitrary n, $\quad k = 2$

$$T_2 f(y; x_0) = f(x_0)$$ deg 0

$$+ \nabla f(x_0) \odot y$$ scalar product deg 1

$m! = 0! \cdots 2! 0! \cdots = 2! = 2$

$m = (0, 0, \ldots, 2, 0, \ldots 0)$

$$+ \sum_{i=1}^{n} \frac{1}{2} \partial^2_{x_i^2} f(x_0)\, y_i^2$$ ①

$$+ \sum_{1 \leq i < j \leq n} 1 \cdot \partial^2_{x_i x_j} f(x_0)\, y_i y_j$$ ②

$\left.\begin{matrix}① \\ ②\end{matrix}\right\}$ deg 2

$(m_1, \ldots, m_n)$ s.t. $m_1 + \ldots + m_n = 2$

① : one $m_i = 2$, others $= 0$

② : $m_i = m_j = 1$, with $i \neq j$, others $m_i' = 0$

<u>Remark:</u> the term of degree 2 is equal to

$$= \frac{1}{2}\, {}^t y \,\mathrm{Hess}_f(x_0)\, y$$

column vector

row vector

vector-matrix products

So

$$\boxed{T_2 f(y; x_0) = f(x_0) + Df(x_0)\cdot y + \frac{1}{2}\, {}^t y \,\mathrm{Hess}_f(x_0)\, y}$$

Property:

$$f \in C^k(U; \mathbb{R})$$

$$x_0 \in U$$

Write $$f(x) = (T_k f)(x - x_0; x_0) + E_k(x)$$

Then

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{|E_k(x)|}{\|x - x_0\|^k} = 0$$

3.8- Critical points

Recall:

Def. $f : U \longrightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$

$x_0 \in U$ is called a **critical point** of f if $\nabla f(x_0) = 0 \in \mathbb{R}^n$

$$\Updownarrow$$

$1 \leq i \leq n \quad \frac{\partial f}{\partial x_i}(x_0) = 0$

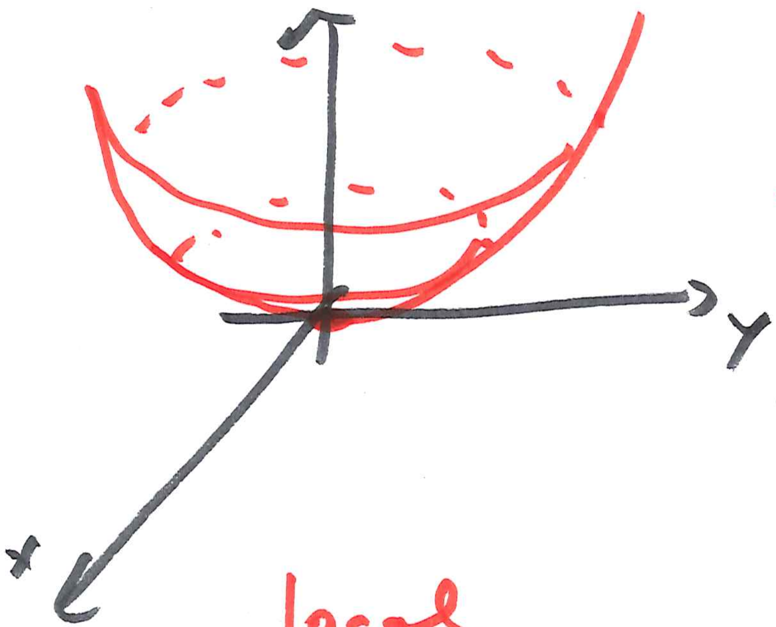
<u>Remark:</u> if x_0 is a point where f has a local max. or minimum, then x_0 is a critical point.

(We say:

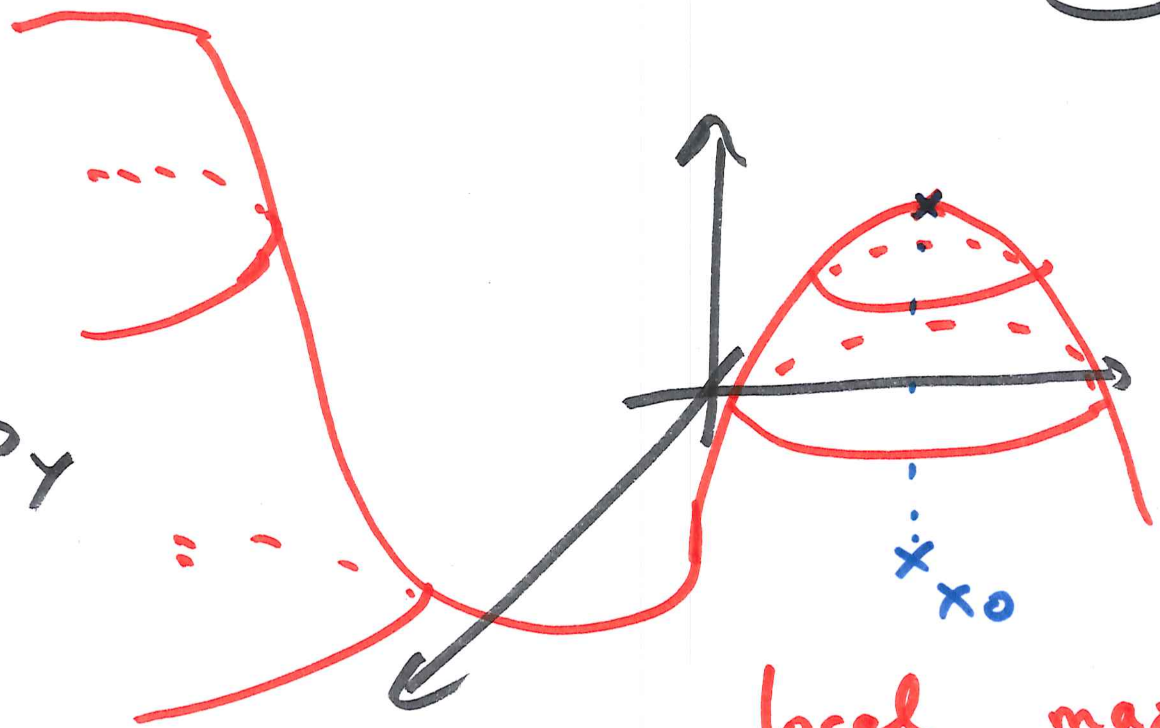
(i) x_0 is a <u>local max.</u> $\iff \exists\, r > 0$, s.t. if $\|x - x_0\| < r$ then $f(x) \leq f(x_0)$

(ii) x_0 is a <u>local min.</u> $\iff \exists\, r > 0$, s.t. if $\|x - x_0\| < r$ then $f(x) \geq f(x_0)$.)

Ex. $n = 2$



local minimum



local max.

Q. given f, find where f is maximal or minimal.

Strategy: in practice, f is defined on a closed bounded set (compact) $\overline{X}$, is continuous, and

$$\overline{X} = \underbrace{U}_{\text{open}} \cup \underbrace{B}_{\text{"boundary"}}$$

with

$$f : U \longrightarrow \mathbb{R}$$

differentiable.

Then:

(i) we know that f has a max./a min. on $\overline{X}$

(ii) a point where f __________ ______ is either

$$\begin{cases} \text{a critical point of } f \text{ in } U \\ \text{a point in } B \end{cases}$$

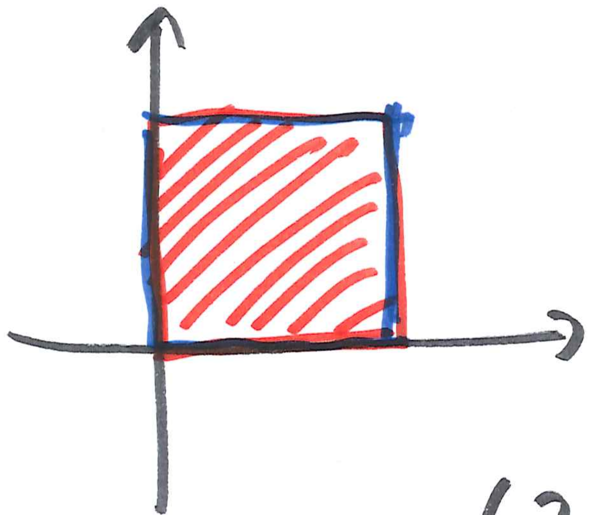
Strategy: find critical points in U, evaluate f at these points and compare with the values on B.

<u>Ex.</u> Max / min

$$f(x,y) = x^2 - 2y^2$$

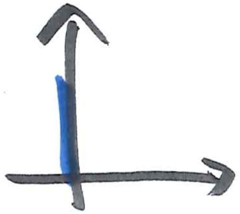
on $\overline{X} = [0,1]^2 \subset \mathbb{R}^2$

$$\overline{X} = \Big(\underbrace{]0,1[\times]0,1[}_{U \text{ open}}\Big) \cup (\text{four segments } \square)$$



$\nabla f = \begin{pmatrix} 2x \\ -4y \end{pmatrix}$, unique possible critical point is $(0,0)$, not in U.

On the boundary,



$0 \le y \le 1$

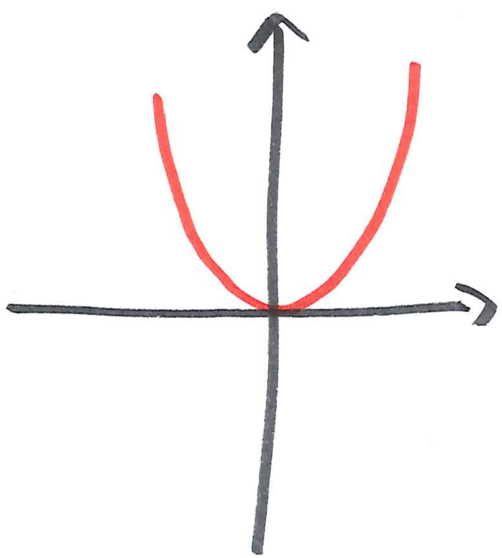
$$f(0,y) = 0 - 2y^2 = -2y^2$$

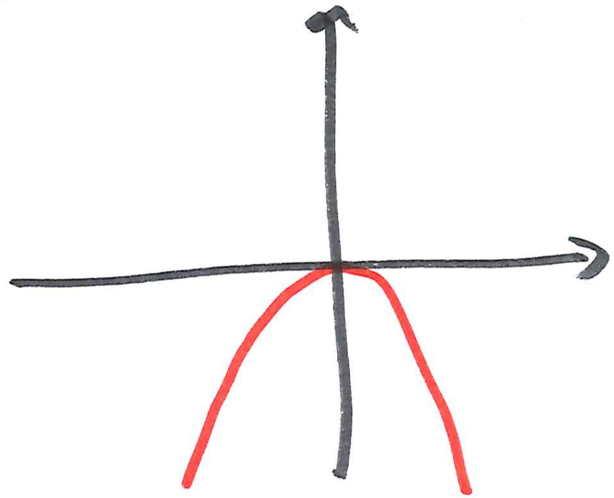
$\hookrightarrow \begin{cases} \max \text{ is } 0 \\ \min \text{ is } -2 \end{cases}$ on that segment

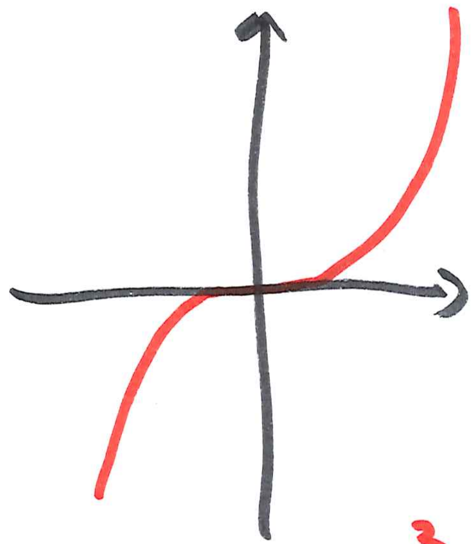
Then repeat for other three segments to find min/max.

Remark. There might be infinitely many critical points!

<u>Q.</u>: how to determine if a critical point is really a local minimum or max?







$f(x) = x^3$

$f''(0) > 0$ $\quad$ $f''(0) < 0$ $\quad$ $f''(0) = 0$

__Def.__ A __non-degenerate__ critical point of $f \in C^2(U; \mathbb{R})$ is a critical point x_0 where

$$\det \mathrm{Hess}_f(x_0) \neq 0.$$

(This is the easiest case, and it happens most of the time.)

x_0 is a non-degenerate critical point

Write the Taylor polynomial of order 2

$$f(x_0) + \underbrace{\cancel{Df(x_0)\cdot y}}_{=0} + \frac{1}{2}\,{}^t y \underbrace{H}_{= \mathrm{Hess}_f(x_0)} y$$

The question is now whether the quadratic term $\frac{1}{2}\,{}^t y H y$ has a local max. or min. at y close to $y_0 = 0$.

From <u>linear algebra</u>: in a suitable basis of $\mathbb{R}^n$ we can write

$$\frac{1}{2}\,{}^t y H y = z_1^2 + \dots + z_p^2 - z_{p+1}^2 - \dots - z_n^2$$

where $(z_1, \dots, z_n)$ are the coord. of y in the new basis.

(Here $0 \leq p \leq n$)

<u>Three cases:</u> (when x_0 non-degenerate)

(1) $\underline{p = n}$ $\Longleftrightarrow$ There is a local <u>minimum</u> at x_0

(H is positive-definite)

(2) $\underline{p = 0}$ $\Longleftrightarrow$ ——— <u>maximum</u> at x_0

(H ——— negative definite)

(3) $\underline{1 \leq p \leq n-1}$ $\Longleftrightarrow$ no local min/max at x_0

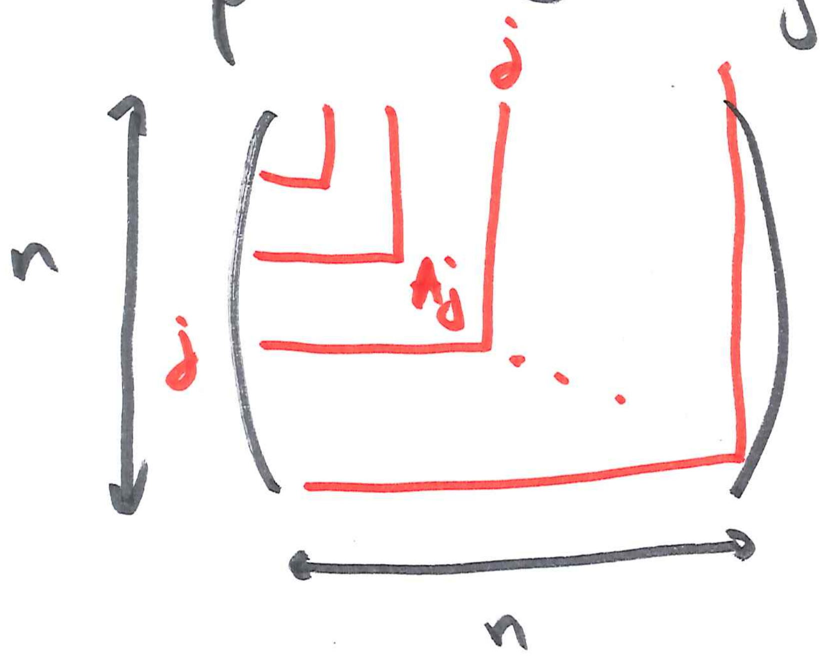
(H is indefinite)

("saddle point" at x_0]

$$\left({}^t y H y = \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_n^2 \right)$$

Criterion (from linear algebra)

(1) A symmetric matrix H is positive definite $\iff$



for $1 \le j \le n$,

$\det(H_j) > 0$

where

$$H_j = (h_{a,b})_{\substack{1 \le a \le j \\ 1 \le b \le j}}$$

Ex.

n = 1 $\quad h_{1,1} > 0$

n = 2 $\quad \begin{pmatrix} a & b \\ b & d \end{pmatrix} : \begin{cases} a > 0 \\ ad - b^2 > 0 \end{cases}$

n = 3 $\quad H = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} : \begin{cases} a > 0 \\ ad - b^2 > 0 \\ \det(H) > 0 \end{cases}$

(2) A matrix H is negative-definite

$\iff$ $-H$ is positive-definite

(**Careful!** $\det(-H_j) = (-1)^j \det(H_j)$!)

(3) Otherwise the matrix is indefinite

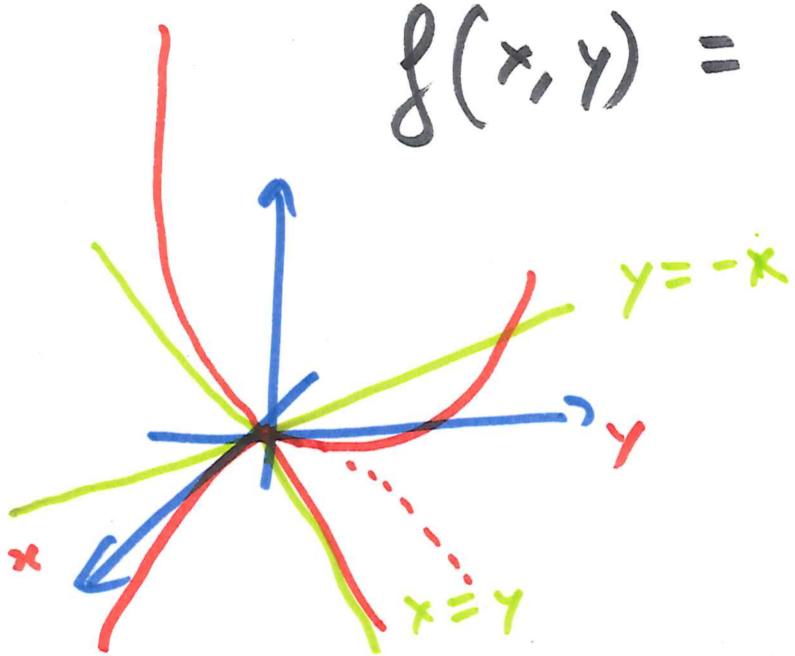
Ex.

(1) $(x_0, y_0) = (0,0)$

$f(x,y) = x^2 + y^2$: local min.

$f(x,y) = -x^2 - y^2$: ——— max.

$f(x,y) = xy$: saddle point



(2)

$$f(x,y) = e^{\cos(x-y)} + x^2$$

$$U =]-4,4[\times]-4,4[$$

One finds three critical points

$$x_1 = (0,0)$$

$$x_2 = (0,\pi)$$

$$x_3 = (0,-\pi)$$

$$H_i = \mathrm{Hess}_g(x_i)$$

$$H_1 = \begin{pmatrix} 2-e & e \\ e & -e \end{pmatrix}$$

$2 - e < 0$

$(2-e)(-e) - e^2 = -2e < 0$

indefinite

$$H_2 = H_3 = \begin{pmatrix} 2+\frac{1}{e} & -\frac{1}{e} \\ -\frac{1}{e} & \frac{1}{e} \end{pmatrix}$$

local minimum $\left\{ \begin{array}{l} 2 + \frac{1}{e} > 0 \\ \frac{2}{e} + \frac{1}{e^2} - \frac{1}{e^2} = \frac{2}{e} > 0 \end{array} \right.$