Examples:

(1) Linear functions are continuous

(2) More generally, polynomials are continuous

(3) Sums of (finitely many) continuous functions are continuous

(4) Product of (finitely many) [real-valued]

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(5)
$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & \mathbb{R}^p \\ \cap & & \cap & & \\ \mathbb{R}^n & & \mathbb{R}^m & & \end{array}$$
: if f & g are continuous, so is $g \circ f$

__Ex.__ $\exp\left(x^3 + \cos(xy) - \sqrt{xyz}\right)$
is continuous on $X = \{(x,y,z) \mid x,y,z > 0\}$

$(\exp(\underbrace{\text{continuous function}}))$

because sum of three continuous functions

polynomials on (conti.)$\circ$(polynomial).

(6) If $f : X \longrightarrow \mathbb{R}$, $g : X \longrightarrow \mathbb{R}$, both continuous __and__ ($g(x) \neq 0$ for all $x \in \mathbb{R}$)
then $f/g : X \longrightarrow \mathbb{R}$ is continuous

Why is (5) true?

$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{R}^p$$

Suppose $x_k \in X$ converges to $x \in X$;

Then $f(x_k) \longrightarrow f(x)$ [f continuous]

so $g(f(x_k)) \longrightarrow g(f(x))$ [g continuous]

$$\begin{array}{ccc} g(f(x_k)) & \longrightarrow & g(f(x)) \\ \| & & \| \\ (g\circ f)(x_k) & \longrightarrow & (g\circ f)(x) \end{array}$$

Warning: suppose $n=2$; if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and $y_0 \in \mathbb{R}$ is fixed, then

$$g(x) = f(x, y_0) : \mathbb{R} \longrightarrow \mathbb{R}$$

is continuous. The converse is false!

Ex. $f(x,y) = \begin{cases} x, & y \geq 0 \\ -x, & y < 0 \end{cases}$

$f(x, y_0) =$ either the function x

or ——— $-x$

which is continuous, but f is not continuous as function of 2 variables

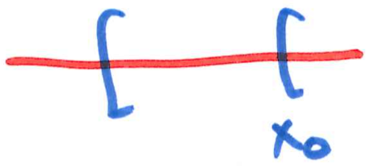
e.g. $f(1,0) = 1$

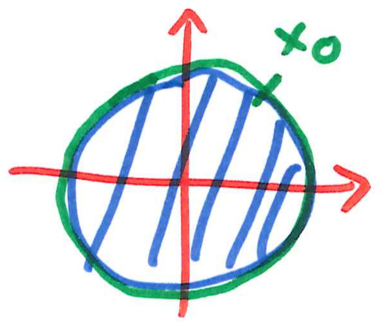
$$(1,0) = \lim_{k\to\infty} \left(1, -\frac{1}{k}\right)$$

and $f\left(1, -\frac{1}{k}\right) = -1.$

<u>Def.</u> $X \subset \mathbb{R}^n$

$x_0 \in \mathbb{R}^n$, not always in X





$X = \{(x,y) \mid x^2+y^2<1\}$, $x_0 = (1,0)$

$$f : X \longrightarrow \mathbb{R}^m$$

We say that $\lim\limits_{x \to x_0} f(x) = Y$

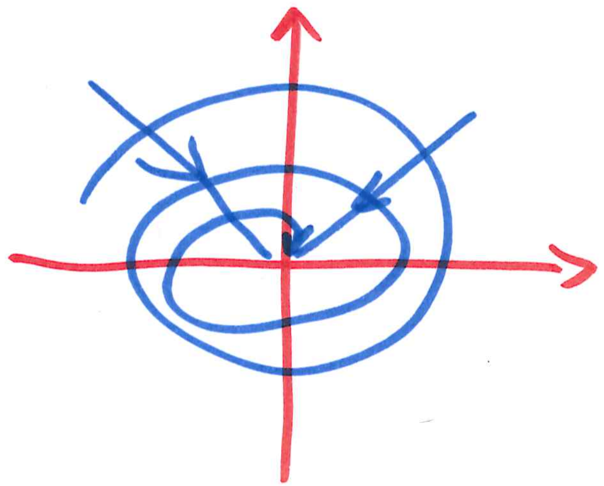
if whenever a sequence (x_k) in X converges to x_0, we have

$$\lim_{k \to \infty} f(x_k) = Y.$$

Ex. (it is important to check <u>all</u> sequences and it may be otherwise that the limit of $f(x_k)$ depends on (x_k))

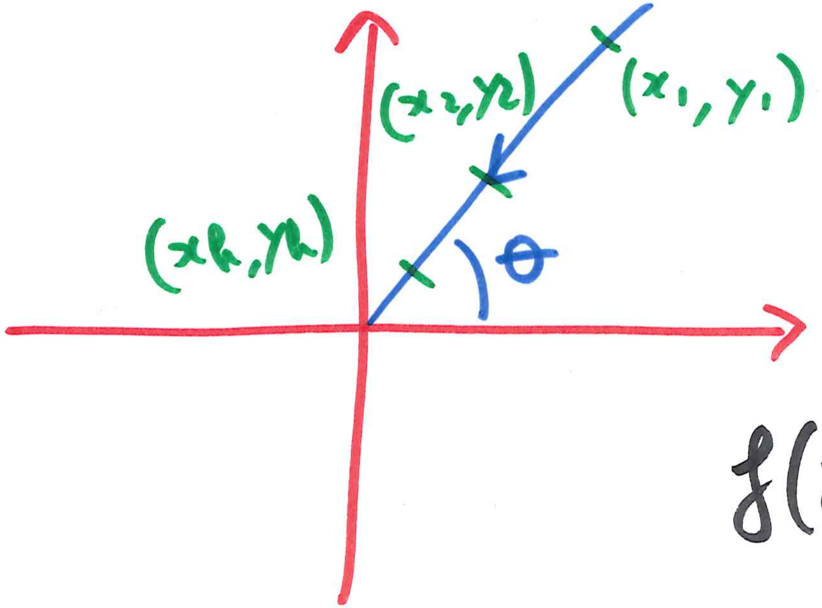
$$f : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \frac{xy}{x^2+y^2}$$

Does $f(x,y)$ have a limit as $(x,y) \longrightarrow 0$?



71

$$f(x,y) = \frac{xy}{x^2+y^2}$$



$$x_k = (r_k \cos\theta, r_k \sin\theta)$$

with $0 < r_k \longrightarrow 0$

$$f(x_k, y_k) = \frac{r_k^2 \cos\theta \sin\theta}{r_k^2}$$

$$= \cos\theta \sin\theta$$

$$\longrightarrow \cos\theta \sin\theta$$

This depends on the angle θ!

One can even find (x_k, y_k) s.t. $f(x_k, y_k)$ has no limit.

Two important facts:

(1) If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous then for any $a \leq b$

$$\{x \in \mathbb{R}^n \mid a \leq f(x) \leq b\} \subset \mathbb{R}^n$$

is **closed**.

(ex. $\{(x,y,z) \mid 0 \leq x^2+y^2+3z^2 \leq 1\}$ is closed

Same is true for the conditions $f(x) \geq a$ or $f(x) \leq b$ (or no condition)

Ex. $\left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \exp\left(x^3 + \cos(xy) - \sqrt{|xyz|}\right) = 1 \right\}$

is closed

Warning! If f continuous, the set

$$X = \{ x \in \mathbb{R}^n \mid a \le f(x) \le b \}$$

is <u>not</u> always compact :

ex. $f(x,y,z) = \cos(xyz)$

$a = -1, \; b = 1$

$\Rightarrow X = \mathbb{R}^n$

(2) <u>Theorem</u> -

$X \subset \mathbb{R}^n$ <u>compact</u> [closed + bounded]

$f: X \longrightarrow \mathbb{R}$ continuous

Then f has (at least) one <u>maximum</u>

x^+ and one <u>minimum</u> x_- in X:

<u>Maximum</u>: for all $x \in X$, $f(x) \leq f(x^+)$

<u>minimum</u> : ______________, $f(x_-) \leq f(x)$

(**Later**: we will see ways to locate x^+ and x_- if f is "differentiable")

3.3 - Partial derivatives

[**Goal**: define for function of $n \geq 2$ variables an analogue of the derivative, which can be used:

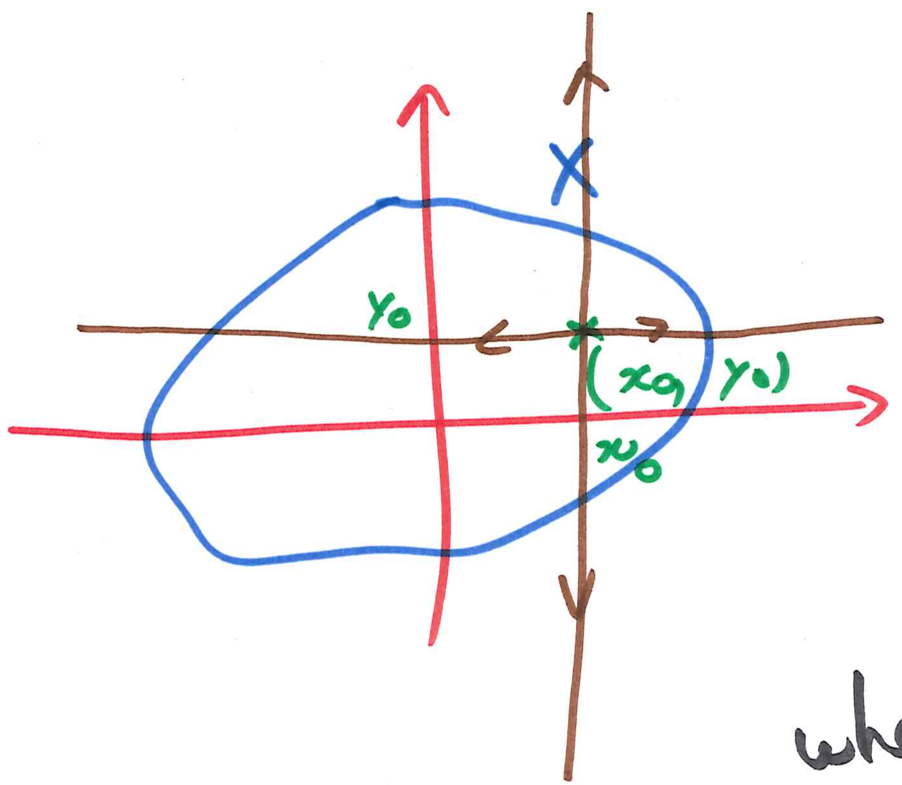
(1) to approximate a function close to a point

(2) to find max/min of a function

(3) to measure the rate of growth of a function]

...

Idea of partial derivatives



$$f : X \longrightarrow \mathbb{R}, \quad X \subset \mathbb{R}^2$$

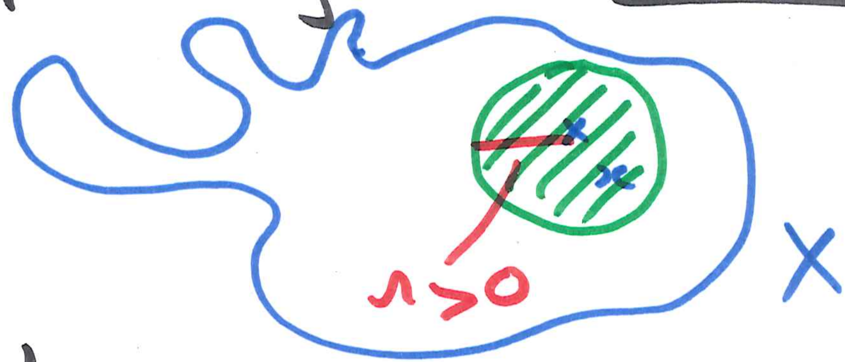
We look at values of f when only one of the coordinates varies, and check if the functions of one variable obtained this way have a derivative.

Def. (Open sets)

$X \subset \mathbb{R}^n$ is called **open** if

$\mathbb{R}^n - X$ [complement] is **closed**

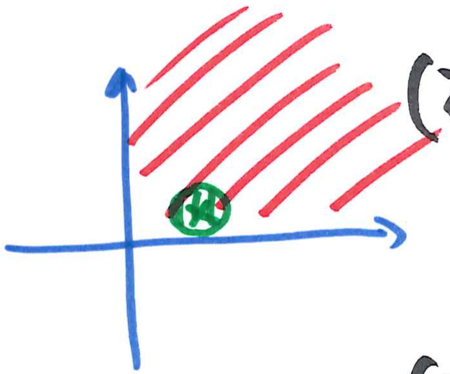
$\Updownarrow$



$$\forall x \in X, \exists r > 0, \{y \in \mathbb{R}^n \mid \|y - x\| < r\} \subset X$$

"One can move from x in any direction while remaining in X"

<u>Ex.</u>



(1) $\mathbb{R}^n$ is open (and closed)

(2) $\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x > 0 \\ y > 0 \end{matrix} \right\}$ is open

(3) $]a_1, b_1[\times \cdots \times]a_n, b_n[$ is open

(4) $I \subset \mathbb{R}$ <u>open interval</u>

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ <u>continuous</u>

$\Rightarrow \{ x \in \mathbb{R}^n \mid f(x) \in I \}$ is <u>open</u>.

(5) $[0, 1[$ not open, not closed

<u>Definition</u> – $X \subset \mathbb{R}^n$ <u>open</u>

$f : X \longrightarrow \mathbb{R}^m$ (arbitrary)

$1 \leq i \leq n$

$(x_{0,1}, \ldots, x_{0,n}) \in X$

f has partial derivative $y \in \mathbb{R}^m$ with respect to the i-th coordinate <u>if</u> the function of one variable

$$g(x) = f(\underbrace{x_{0,1}, \ldots, x_{0,i-1}}_{\text{fixed}}, x, \underbrace{x_{0,i+1}, \ldots, x_{0,n}}_{\text{fixed}})$$

has derivative $g'(x_{0,i}) = y$.

<u>Recall:</u> for $g : \mathbb{R} \longrightarrow \mathbb{R}^m$, given by

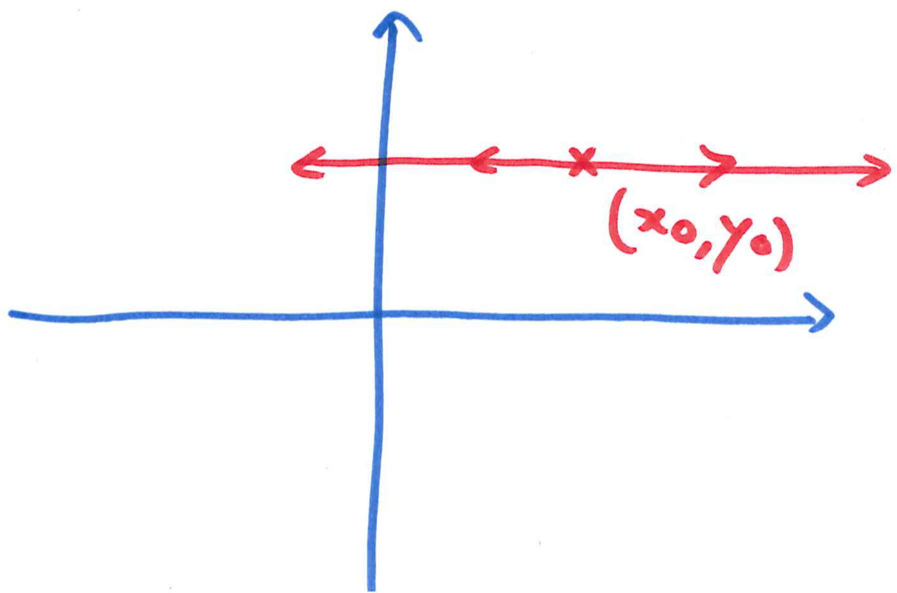
$$g(x) = (g_1(x), \ldots, g_m(x))$$

we say that g ~~is~~ ~~dif~~ has derivative

$$y = (y_1, \ldots, y_m)$$

at x_0 if

$$g_1'(x_0) = y_1, \ldots, g_m'(x_0) = y_m$$



$g(x) = f(x, y_0)$, x close to x_0

Notation:

$$\frac{\partial f}{\partial x_i}(x_0)$$

"variable"

"value"

$$\partial_{x_i} f\,(x_0)$$

$$\partial_i f\,(x_0)$$

e.g.

$$\frac{\partial f}{\partial x}(x, y)$$

value

variable

Ambiguity!

<u>In practice:</u>

$$f(x_1, \dots, x_n) = \dots \text{ some complicated formula} \dots$$

$$\frac{\partial f}{\partial x_1} = \text{derivative when we think that } x_2, x_3, \dots \text{ are just parameters}$$

<u>Ex.</u>

$$f(m, c) = mc^2$$

$$\frac{\partial f}{\partial m} = c^2 \quad , \quad \frac{\partial f}{\partial c} = 2mc$$

This is as easy as derivatives with one variable...

The usual rules for derivatives apply (<u>except</u>! the Chain Rule):

(1) $$\frac{\partial (f+g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_i}$$

(2) $$\frac{\partial (fg)}{\partial x_i} = f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i}$$

(3) if $g(x)$ is not zero when x is close to x_0,

$$\frac{\partial (f/g)}{\partial x_i} = \frac{\partial f/\partial x_i \; g - \frac{\partial g}{\partial x_i} f}{g^2}$$

Examples:

(1) Polynomials (incl. linear function) have partial derivatives: if

$$f(x_1, \dots, x_n) = x_1^{m_1} \cdots x_n^{m_n}$$

$$\Rightarrow \frac{\partial f}{\partial x_i} = m_i \, x_1^{m_1} \cdots x_{i-1}^{m_{i-1}} x_i^{m_i - 1} x_{i+1}^{m_{i+1}} \cdots x_n^{m_n}$$

(e.g. $\frac{\partial}{\partial z}(x^2 y^3 z^7) = 7 x^2 y^3 z^6$)

(2) $f(x) = Ax$ (linear case)

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (A matrix m rows, n columns)

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m$$

$$\frac{\partial f}{\partial x_i} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} \in \mathbb{R}^m$$

function of x

<u>constant</u>
<u>vector</u>

<u>Definition</u> -

$X \subset \mathbb{R}^n$ <u>open</u>

$f : X \longrightarrow \mathbb{R}^m$ function, $f = (f_1, \dots, f_m)$

If f has all partial derivatives at $x_0 \in X$ then the <u>Jacobi matrix</u> $J_f(x_0)$ is the matrix with m rows n columns

$$J_f(x_0) = \left(\frac{\partial f_j}{\partial x_i}(x_0) \right)_{\substack{1 \le j \le m \\ 1 \le i \le n}}$$

Ex.

(1) $f(x) = Ax$, $\quad A$ matrix with $\begin{cases} m \text{ rows} \\ n \text{ columns} \end{cases}$

Then

$$J_f(x) = A$$

for all x $\quad$ (constant matrix)

89

(2) $$f(x,y) = \begin{pmatrix} e^{x} + xy \\ -\cos(xy) \\ 1 + \dfrac{xy}{x^2+y^2} \end{pmatrix} \qquad (x,y) \neq (0,0)$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$J_f(x,y) = \begin{array}{cc} 1 & 2 \\ \left(\begin{array}{c|c} e^{x} + y & x \\ +y\sin(xy) & +x\sin(xy) \\ ? & ? \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \uparrow & \uparrow \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{array}$$