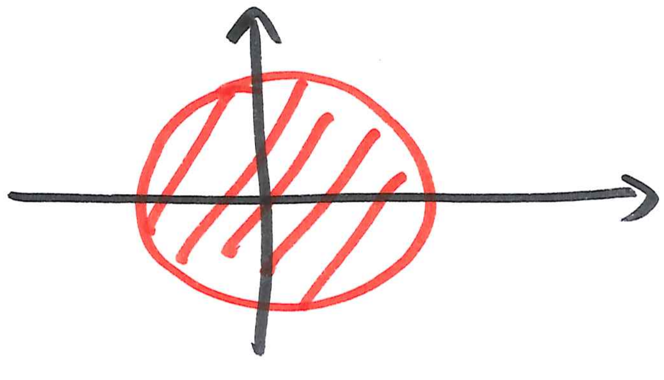
__In practice__ : we __know__ in advance what sets Y we want to integrate on and what X corresponds to Y.

One uses polar coordinates for instance for:

(1) __Disc centered at 0__



Radius $R > 0$

$$Y = \{ (x,y) \mid x^2 + y^2 \leq R^2 \}$$

$$X = [0, R] \times [0, 2\pi]$$

Here taking $Y_0 = \{(x,y) \mid 0 < x^2+y^2 < R^2\}$ 265

$X_0 =]0, R[\times]0, 2\pi[$

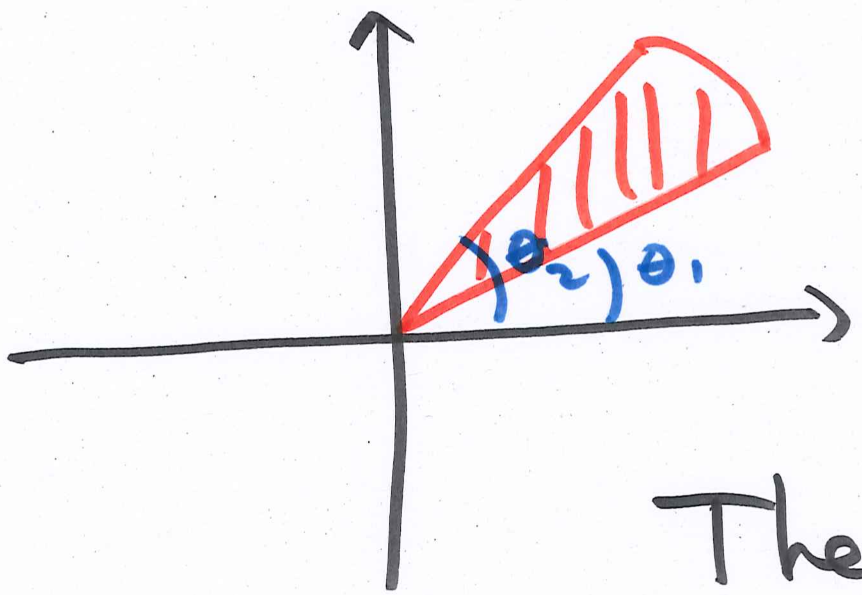
the conditions are satisfied so:

for <u>any</u> continuous $f : Y \longrightarrow \mathbb{R}$

$$\boxed{\int_Y f(x,y)\, dx\, dy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr}$$

Y = disc of radius R

(2) Sector



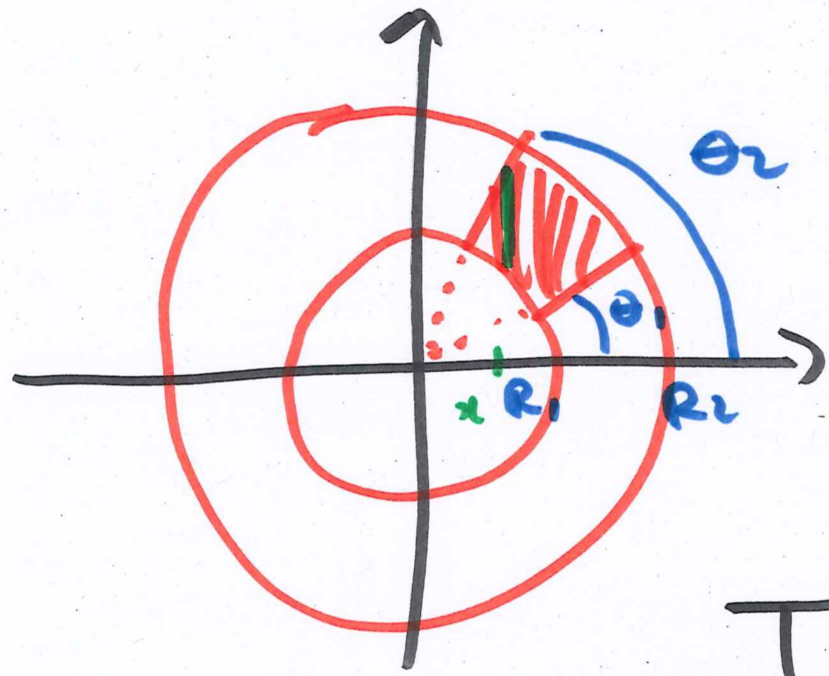
$$Y = \{(x,y) \mid x^2+y^2 \leq R^2, \; \theta_1 \leq \theta \leq \theta_2\}$$

$$X = [0,R] \times [\theta_1, \theta_2]$$

The change of variable applies:

$$\int_Y f(x,y)\,dx\,dy = \int_0^R \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\, r \; d\theta\, dr$$

(3) **Annulus**



$$Y = \{(x,y) \mid R_1^2 \le x^2 + y^2 \le R_2^2,\ \theta_1 \le \theta \le \theta_2\}$$

$$X = [R_1, R_2] \times [\theta_1, \theta_2]$$

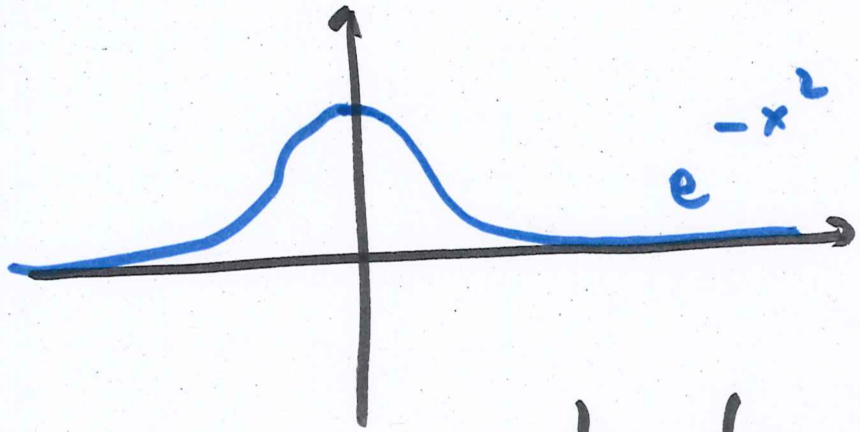
The formula applies:

$$\int_Y f(x,y) = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} \cdots\cdots r \, d\theta \, dr$$

Example:

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

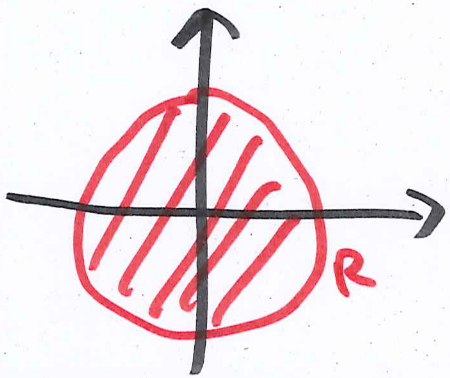
("Gaussian integral")



We start by computing

$$\int_{\gamma_R} e^{-x^2-y^2}\,dx\,dy$$

where γ_R = disc of radius R



Polar coordinates:

$$\int_{Y_R} e^{-x^2-y^2}\,dx\,dy = \int_0^R \int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr$$

$$= 2\pi \int_0^R r e^{-r^2}\, dr$$

Fubini

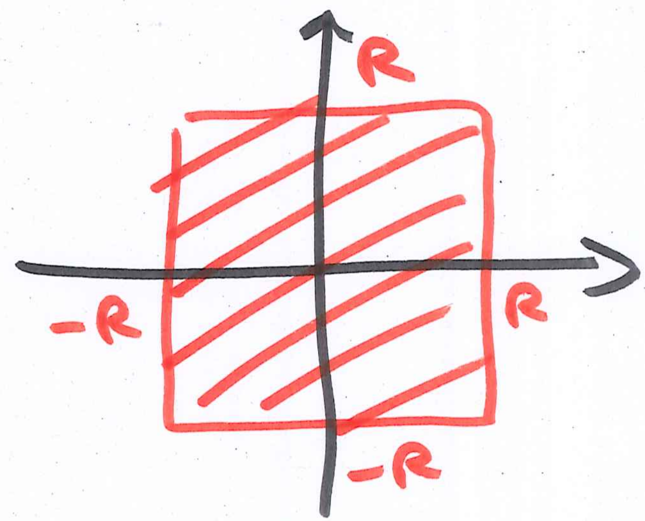
$r^2 = u$
$2r\,dr = du$

$$= \pi \int_0^{R^2} e^{-u}\, du = \pi\left(1 - e^{-R^2}\right)$$

On the other hand

$$\int_{[-R,R]^2} e^{-x^2-y^2}\,dx\,dy$$

$$\overset{\text{Fubini}}{=} \left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2$$



Now let $R \longrightarrow +\infty$: this quantity converges to

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2$$

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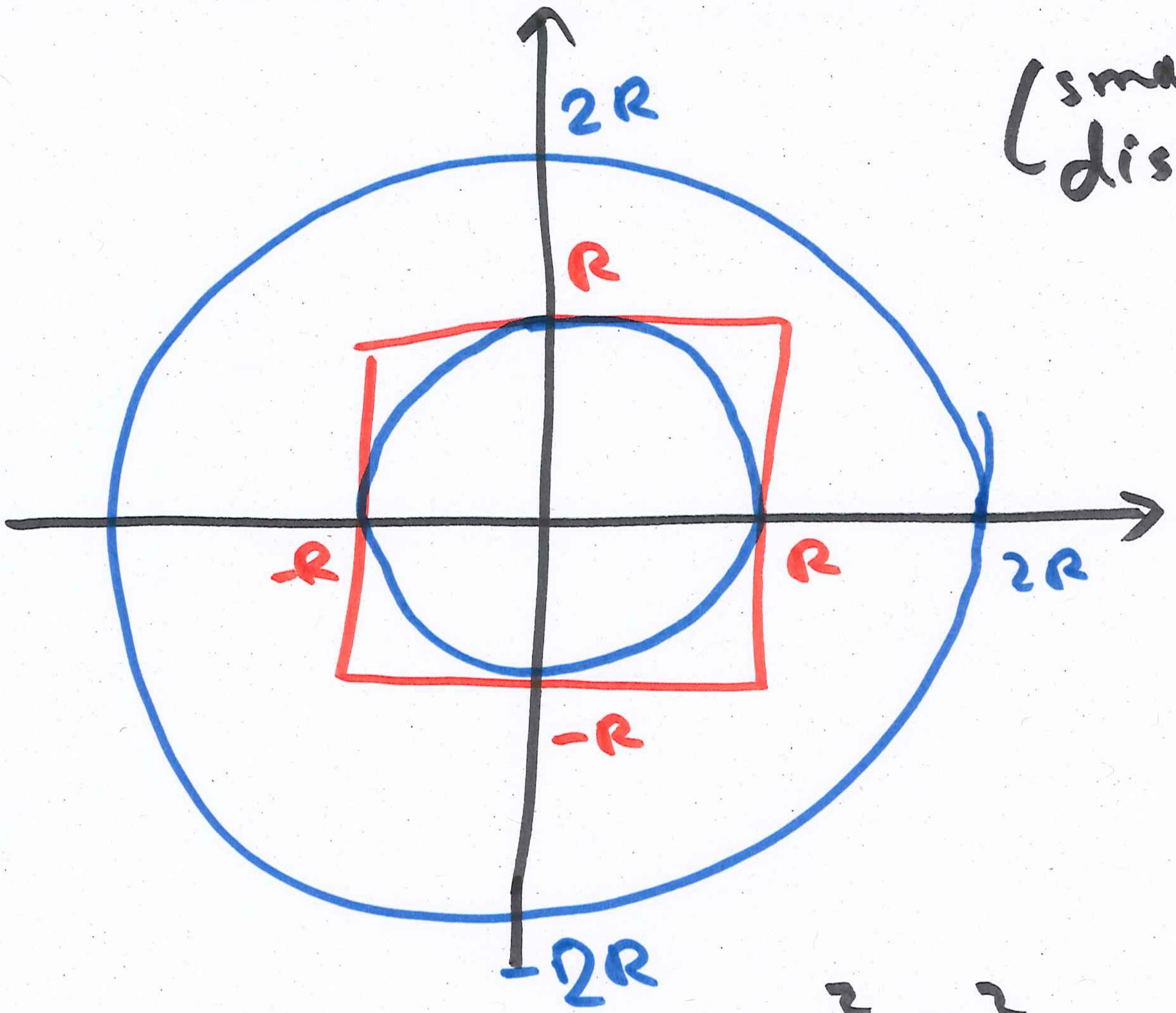

$\begin{pmatrix}\text{small}\\ \text{disc}\end{pmatrix} \subset (\text{square}) \subset \begin{pmatrix}\text{big}\\ \text{disc}\end{pmatrix}$

so since $\left(e^{-x^2-y^2} \geq 0\right)$ we get

$$\int_{\begin{pmatrix}\text{disc radius}\\ R\end{pmatrix}} e^{-x^2-y^2}\,dx\,dy \leq \left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2 \leq \int_{(\text{radius } 2R)} e^{-x^2-y^2}\,dx\,dy$$

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so $$\pi\left(1-e^{-R^2}\right) \leq \left(\int_{-R}^{R} e^{-x^2}dx\right)^2 \leq \pi\left(1-e^{-4R^2}\right)$$

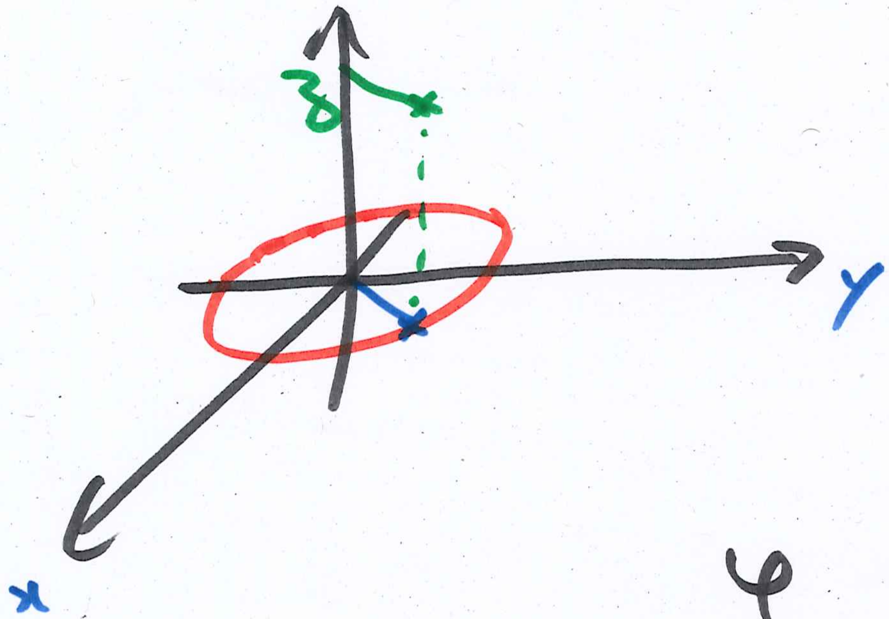
<u>$R \to +\infty$:</u> $$\pi \leq \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 \leq \pi$$

<u>Conclusion:</u>

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}}$$

Other changes of variables :

① $\underline{n=3}$: cylindrical coordinates



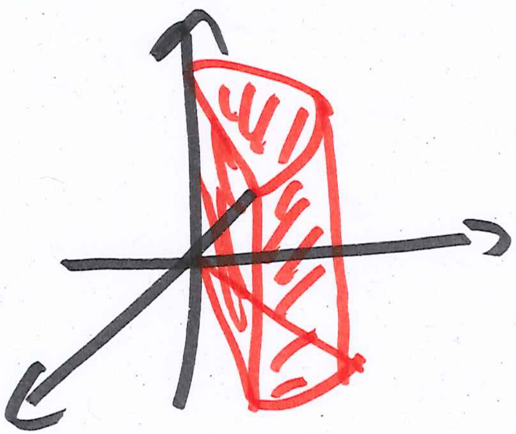
$$(x, y, z)$$
$$\big\downarrow$$
$$(r, \theta, z)$$

$$\varphi \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$(r > 0, \ \theta \in]0, 2\pi[, \ z \in \mathbb{R})$

so $|\det J_\varphi(r, \theta, z)| = r$

Useful for integration over <u>cylinders</u>, where the intersection with (x,y) plane is adapted to polar coordinates:



<u>ex</u>

$$0 \le r \le R$$
$$\theta_1 \le \theta \le \theta_2$$
$$0 \le z \le Z$$

② <u>$n = 3$:</u> spherical coordinates

(for integration over spheres)

(275)

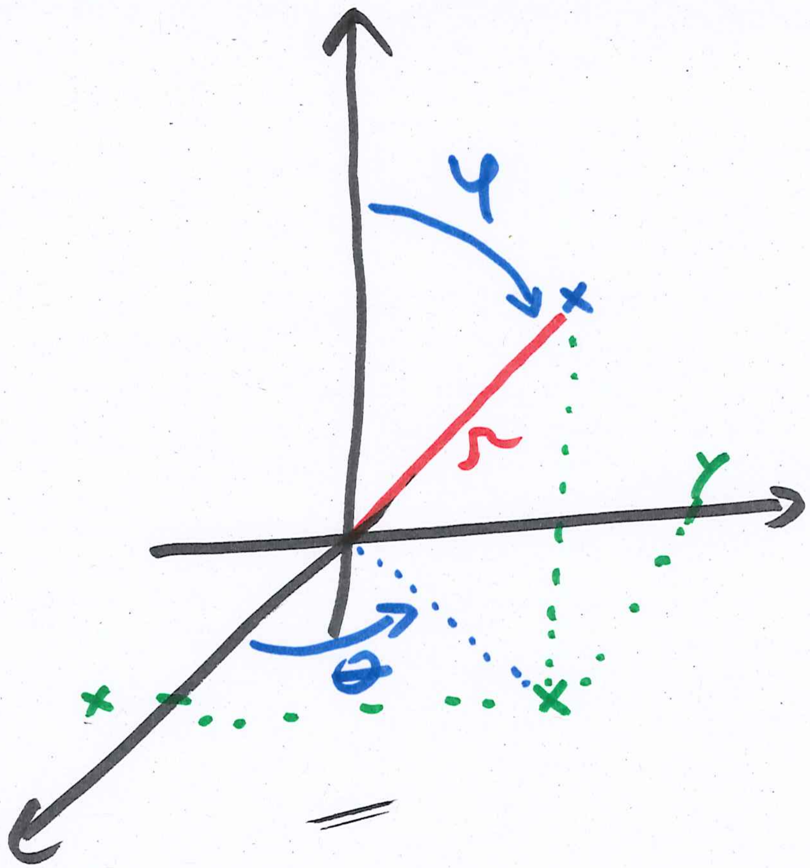
(x, y, z)

$\updownarrow$

r = distance to O

θ = polar angle for $(x, y, 0)$

φ = "azimuthal" angle



$$\underset{r}{[0, +\infty[} \times \underset{\theta}{[0, 2\pi]} \times \underset{\varphi}{[0, \pi]}$$

$$\begin{cases} x = r\cos(\theta)\sin(\varphi) \\ y = r\sin(\theta)\sin(\varphi) \\ z = r\cos(\varphi) \end{cases}$$

One computes:

$$\left|\det J_{\substack{\text{spherical}\\ \text{coord}}}(r,\theta,\varphi)\right| = r^2\sin(\varphi)$$

(p^o55 of the script).

Integrating over a sphere:

$$Y = \{(x,y,z) \mid x^2+y^2+z^2 \leq R^2\}$$

$$X = \{(r, \theta, \varphi) \mid 0 \le r \le R\}$$
$$= [0, R] \times [0, 2\pi] \times [0, \pi]$$

Change of variable is:

$$\boxed{\int_Y f(x,y,z)\, dx\, dy\, dz = \int_{r=0}^{R} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} f(r, \theta, \varphi)\, r^2 \sin(\varphi)\, d\varphi\, d\theta\, dr}$$

__Ex.__ $f = 1$, $R = 1$

Vol(sphere of radius 1 in $\mathbb{R}^3$) =

$$= \int_0^1 \int_0^{2\pi} \int_0^{\cancel{2}\pi} r^2 \sin(\varphi) \, dr \, d\theta \, d\varphi \quad 2 \to 8$$

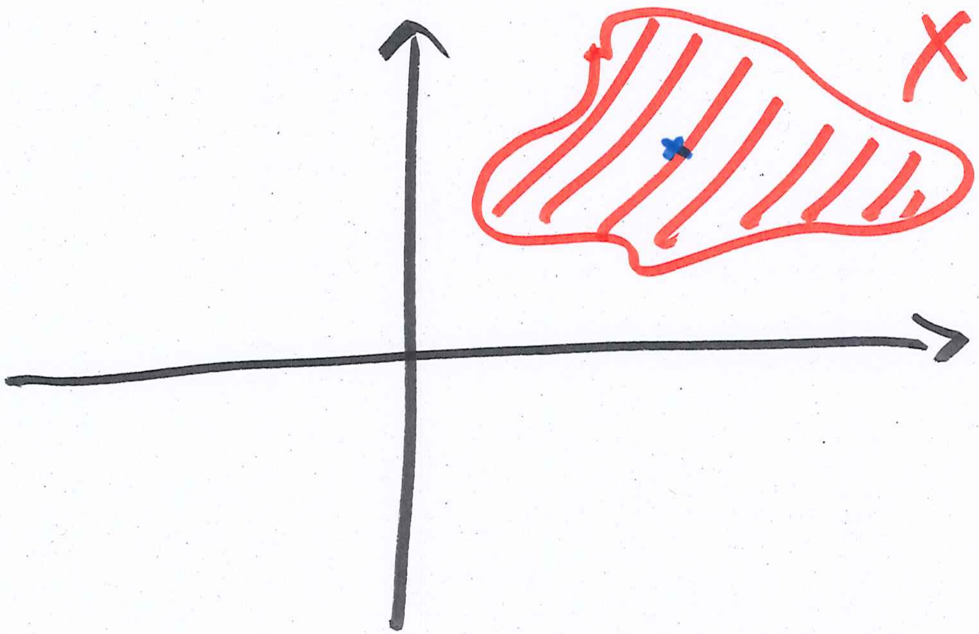
$$= \left(\int_0^1 r^2 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin(\varphi) \, d\varphi \right)$$

$$= \frac{1}{3} \times 2\pi \times \underbrace{\left[\cos \varphi \right]_0^{\pi}}_{1-(-1)=2}$$

$$= \frac{4\pi}{3}$$

4.5. Other geometric applications

① Center of mass



$X \subset \mathbb{R}^n$
bounded
closed

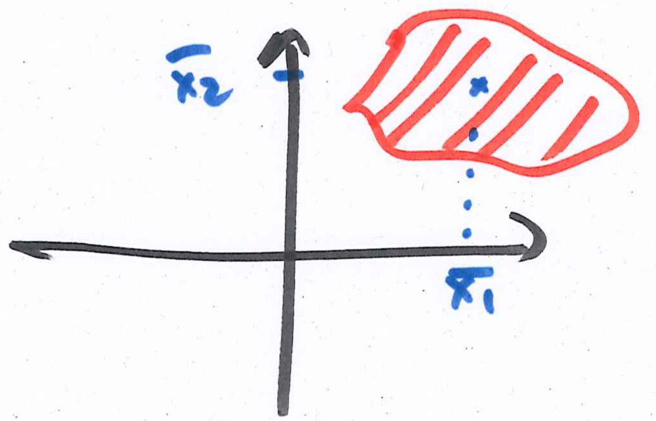
Center of mass of X : point where X is "perfectly balanced"

Coordinates of the center of mass:

$$\overline{x} = (\overline{x_1}, \dots, \overline{x_n}) \in \mathbb{R}^n$$

$$\overline{x_i} = \frac{1}{\mathrm{vol}(X)} \int_X x_i \, dx_1 \dots dx_n$$

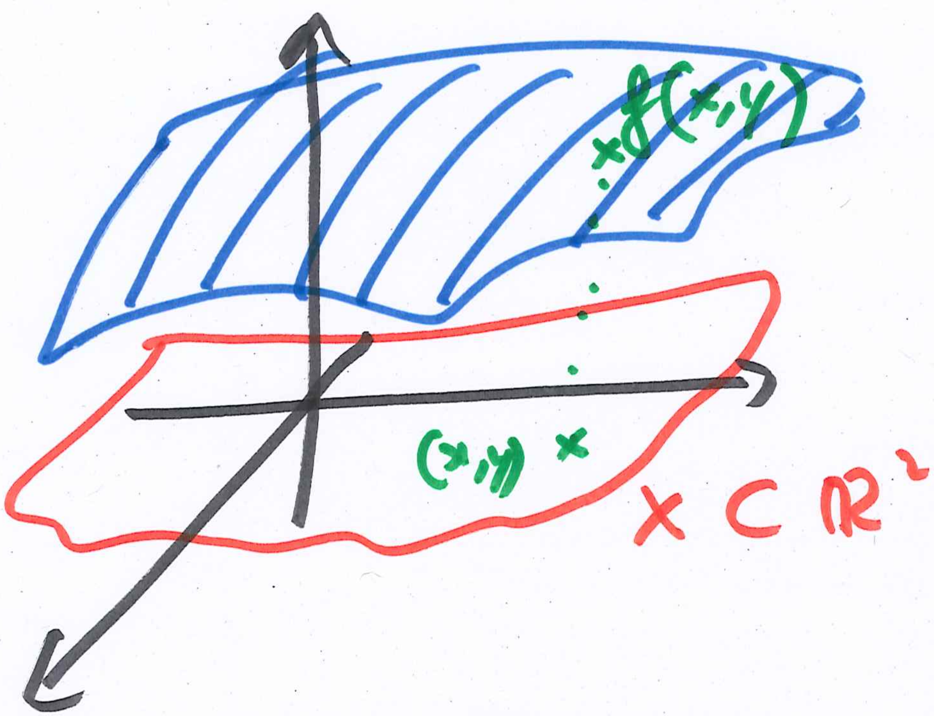
("average of the x_i-coordinate")



2 Surface area

$n = 3$

$S =$ "surface" bounded and closed in $\mathbb{R}^3$



Area of $S = ?$

When S is a graph:

$$S = \{(x, y, z) \mid (x, y) \in X \subset \mathbb{R}^2, \; z = f(x,y)\}$$

for a function $f : X \longrightarrow \mathbb{R}$ C^1

then

$$\boxed{\text{Area}(S) = \int_X \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2}\, dx\,dy}$$

$X \subset \mathbb{R}^2$

<u>Ex.</u> Area of a sphere of radius 1 in $\mathbb{R}^3$

$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$

Check:

$$\boxed{\text{Area}(S) = 4\pi}$$

(Example 4.5.1 (2))

4.6 – The Green formula

This is the simplest of many formulas that relate an n-dim. integral with an $(n-1)$-dim. one.

Green formula:

$n=2$

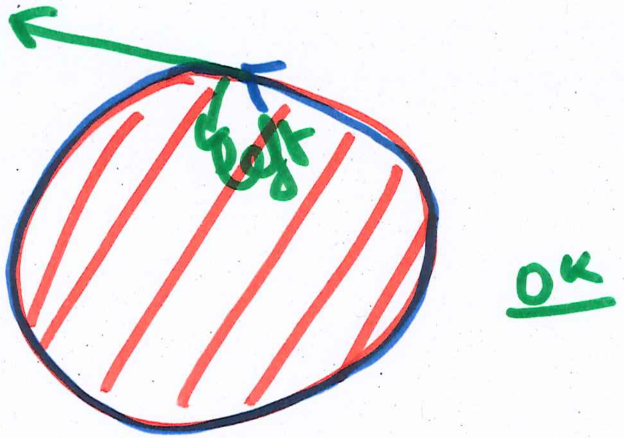
(2-dim. integral) = (line integral)

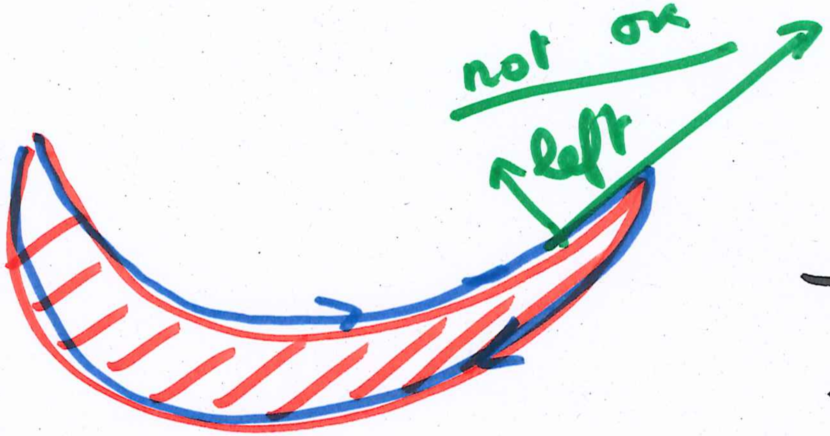
Formula: $X \subset \mathbb{R}^2$ closed bounded

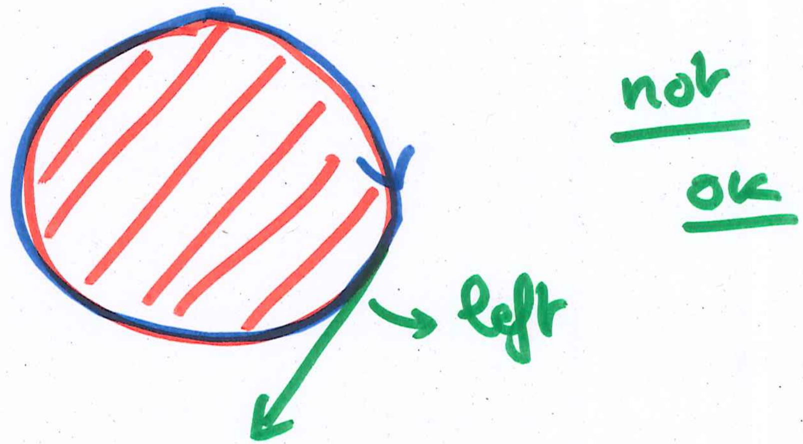
∂X = boundary of X

Hyp. ① ∂X is a simple closed parameterized curve

② X is always on the left-hand side of a tangent vector to the boundary







<u>Then</u>: for any C^1 vector field $f = (f_1, f_2)$ on $\mathbb{R}^2$, we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy = \int_{\partial X} f \cdot d\vec{s}$$

Remark: the choice of orientation is very important since the line integral changes sign if orientation is reversed

Example: f is conservative

$\Longrightarrow$ formula is correct but says that $0 = 0$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$$

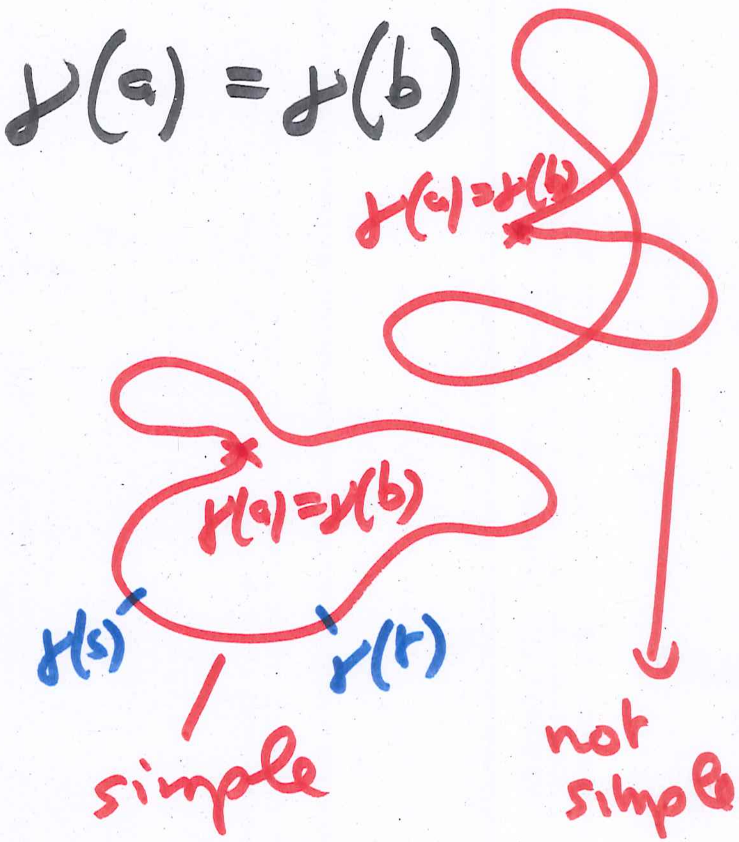
line integral over a closed curve is 0

Recall: $\gamma : [a,b] \longrightarrow \mathbb{R}^n$

parameterized curve

is :
- **closed**: if $\gamma(a) = \gamma(b)$
- **simple**: if

There are not
$$a < s < t < b$$
s.t. $\gamma(s) = \gamma(t)$



<u>Generalization</u> : multiple boundary parts

∂X = union of finitely many γ_i which are simple closed curves

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \sum_i \int_{\gamma_i} f \cdot d\vec{s}$$

when each γ_i has "X on the left"

Example:

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy$$

$$= \int_\gamma f \cdot d\vec{s}$$

1. Given a function

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

we may recognize that

$$g = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

for some vector field f.

Then we can replace a 2-dim. integral of g with a line integral.

Ex. $g = 1$ so $\int_X g \, dx\,dy = \text{Area}(X)$

We can take for instance

$$f(x,y) = (0, x)$$

$$\Rightarrow \boxed{\text{Area}(X) = \int_{\partial X} (0, x) \cdot d\vec{s}}$$