3.9 - Lagrange multipliers

Q. maximize $f(x, y, z)$

for $(x, y, z) \in \mathbb{R}^3$ such that

$$g(x, y, z) = 0.$$

Ex. maximize $f(x, y, z)$

on $x^2 + y^2 + z^2 = 1$

3.10 - Inverse / implicit function theorems

$(x, y) \in \mathbb{R}^2$ satisfy

$$g(x,y) = 0 \quad , \quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Can we write y as a function of x?

Ex. $$x^2 + xy + e^{xy} = 0$$

Chapter 4

Integration in $\mathbb{R}^n$

Goals:

- Define the integral of a function $f: \underset{\mathbb{R}^n}{\overset{}{U}} \longrightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$
- Relations between such integrals with different n. (Fundamental th. of calculus)

4.1. <u>Line integrals</u> / <u>path integrals</u>

<u>Idea:</u> integrate a function along a curve in $\mathbb{R}^n$

<u>Def.</u> (Parameterized curve / path)

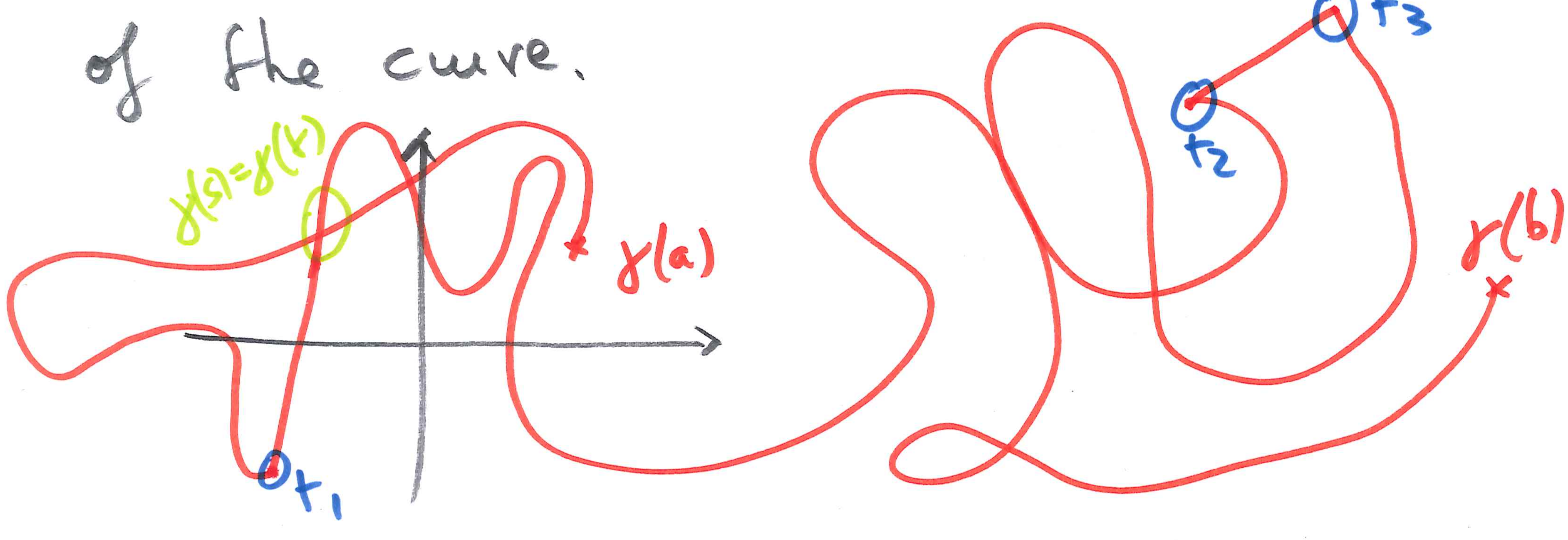
$n \geq 2$

A <u>parameterized curve in $\mathbb{R}^n$</u> is a continuous function

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n$$

<u>piecewise C^1</u>, which means:
there exist
$t_0 = a < t_1 < \dots < t_n = b$
s.t. $f:]t_i, t_{i+1}[\longrightarrow \mathbb{R}^n$ is C^1.

We call $\gamma(a)$, $\gamma(b)$ the <u>extremities</u> of the curve.



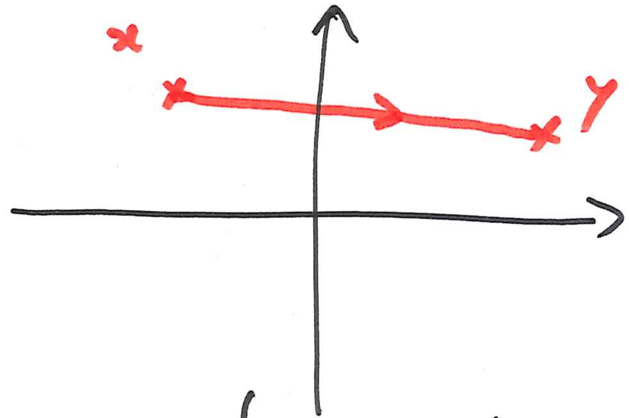
185 **Note:** we can have $\gamma(s) = \gamma(t)$ for some $s \neq t$.

Ex. (1) $\gamma(t) = (x_0, y_0)$ for all t is allowed

(2) Let x, y be arbitrary points of $\mathbb{R}^n$. Define

$$\gamma(t) = (1-t)x + ty, \quad t \in [0,1]$$

This is a parameterized curve; it goes from $\gamma(0) = x$ to $\gamma(1) = y$; the image of γ is the <u>line segment</u> in $\mathbb{R}^n$ joining x to y.

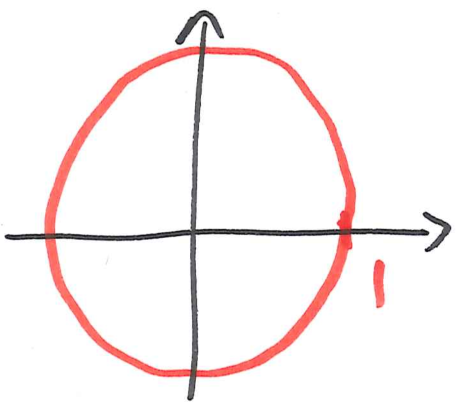


(3) There are many parameterized curves with the same geometric image in $\mathbb{R}^n$:

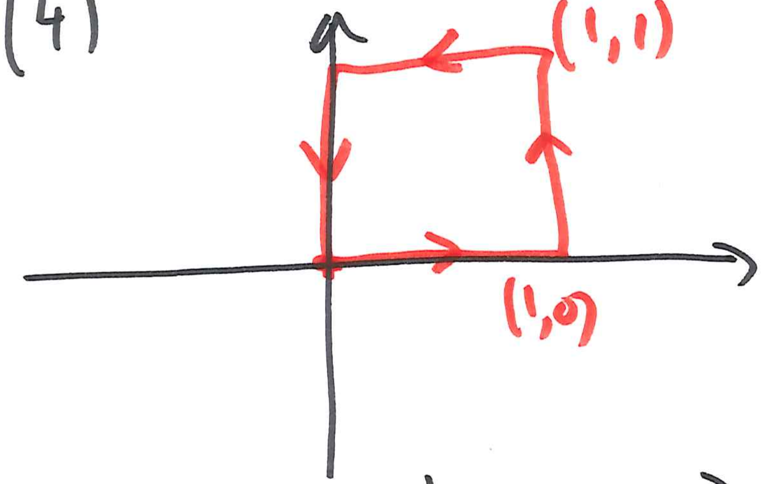
$k \geq 1$

$$\gamma_k(t) = \left(\cos(2\pi t^k), \sin(2\pi t^k)\right) \quad t \in [0,1]$$

has image the circle of radius, for <u>all</u> k.



(4)



(A parameterized curve is <u>closed</u> if $\gamma(a)=\gamma(b)$)

$$\gamma(t) = \begin{cases} (4t,\ 0) & 0 \le t \le 1/4 \\ (1, \cancel{3}4(t - \frac{1}{4})) = (1,\ 4t-1) & \frac{1}{4} \le t \le \frac{1}{2} \\ (-4(t-\frac{1}{2})+1,\ 1), & \frac{1}{2} \le t \le \frac{3}{4} \\ (0,\ -4(t-\frac{3}{4})+1), & 3/4 \le t \le 1 \end{cases}$$

has 4 "corners" where γ is not C^1.

Definition. $n \geq 2$

$\gamma : [a,b] \longrightarrow \mathbb{R}^n$ parameterized curve

$X \subset \mathbb{R}^n$ containing the image

$f : X \longrightarrow \mathbb{R}^n$ continuous ("vector field")

$$\int_\gamma f(s) \cdot d\vec{s} \overset{\text{def}}{=} \int_a^b \underbrace{f(\gamma(t))}_{\in \mathbb{R}^n} \cdot \underbrace{\gamma'(t)}_{\mathbb{R}^n} \, dt \quad (\in \mathbb{R})$$

scalar product

"integral of f along γ"

Concretely: if $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

then $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$

and $f(x) = (f_1(x), \dots, f_n(x))$

then
$$\int_\gamma f(s) \cdot d\vec{s} = \int_a^b \sum_{i=1}^n \underbrace{f_i(\gamma(t))}_{\text{continuous}} \underbrace{\gamma_i'(t)}_{\substack{\text{continuous} \\ \text{(possibly except at} \\ \text{finitely many corners)}}} \, dt$$

$\longrightarrow$ the integral exists and is a real number.

<u>Most important fact:</u>

"independant of parameterization of the curve"

<u>Def.</u> $\gamma : [a,b] \longrightarrow \mathbb{R}^n$

Given $\sigma : [c,d] \longrightarrow [a,b]$

which is continuous, strictly increasing, and bijective, C^1 on $]c,d[$

Then $\gamma \circ \sigma : [c,d] \longrightarrow \mathbb{R}^n$

is called a oriented <u>reparameterization</u> of γ

Th. (4.1.5)

If γ_1, γ_2 are curves with γ_2 a reparameterization of γ_1, Then for any $f : X \longrightarrow \mathbb{R}^n$

$$\int_{\gamma_1} f(s) \cdot d\vec{s} = \int_{\gamma_2} f(s) \cdot d\vec{s}$$

(So we can try to find the best possible parameterization !)