Analysis II

INFK

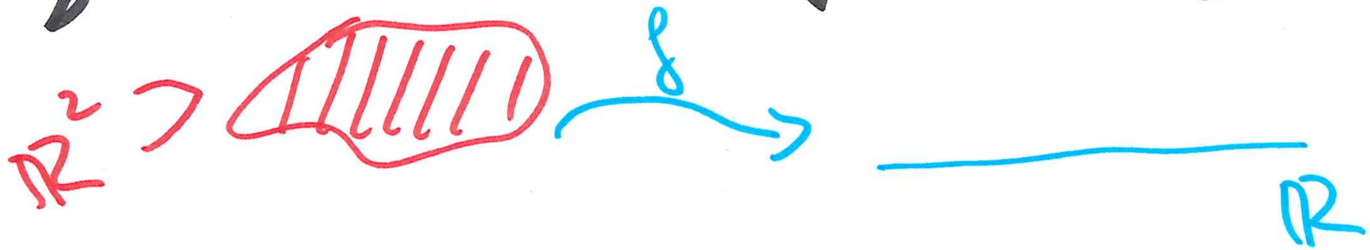
HS 2019

Outline

1 - Differential equations

2 - Differentiation } for more than one variable

3 - Integration



Chapter 1

Reminders

See the script of Prof. Burger for all definitions.
Small addition: functions defined on $I \subset \mathbb{R}$ with

values in $\mathbb{R}^d$:

<u>ex.</u> $f(x) = (\cos(x), \sin(x), 2x+1)$

$\in \mathbb{R}^3$

$f : \mathbb{R} \longrightarrow \mathbb{R}^3$

In general:

$f_i : I \longrightarrow \mathbb{R}$

$f(x) = (f_1(x), \ldots, f_d(x))$

Say: f is continuous $\iff$ <u>each</u> f_i is

differentiable

If f is differentiable, define

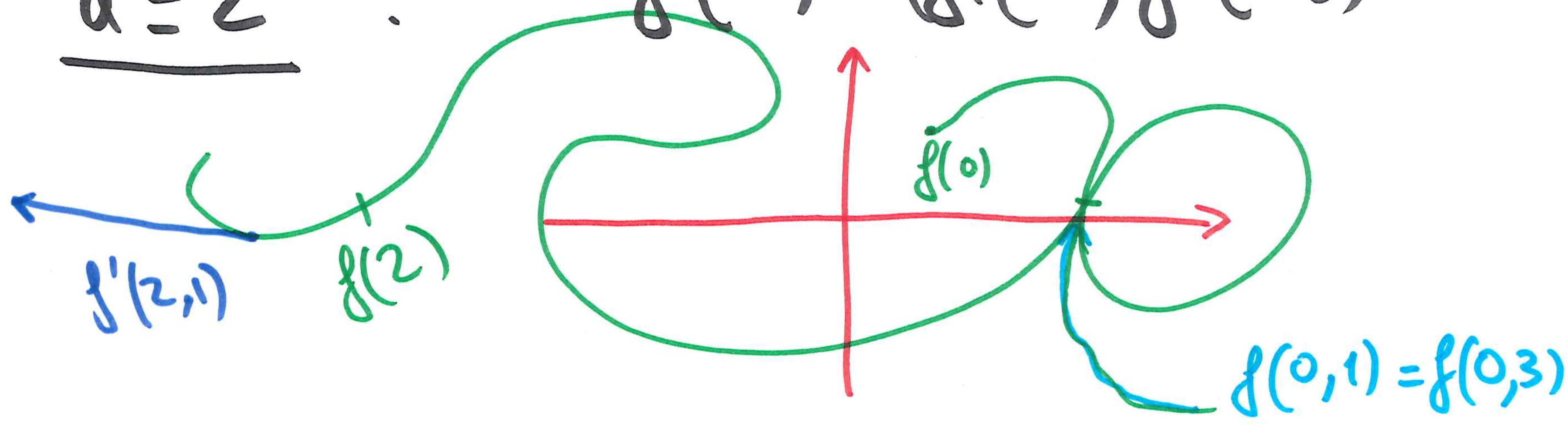
$$f'(x) = (f_1'(x), \dots, f_d'(x))$$

ex.

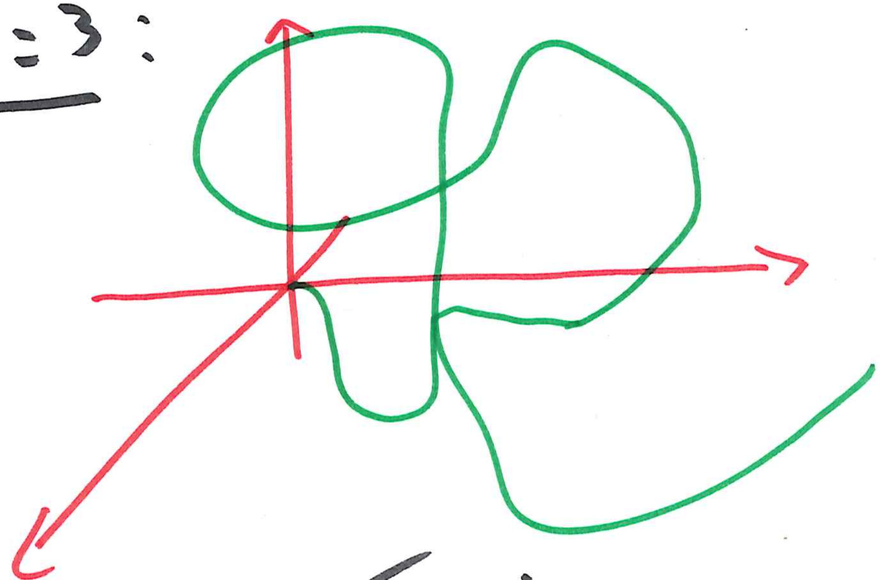
$$f'(x) = (-\sin(x), \cos(x), 2)$$

Visualization:

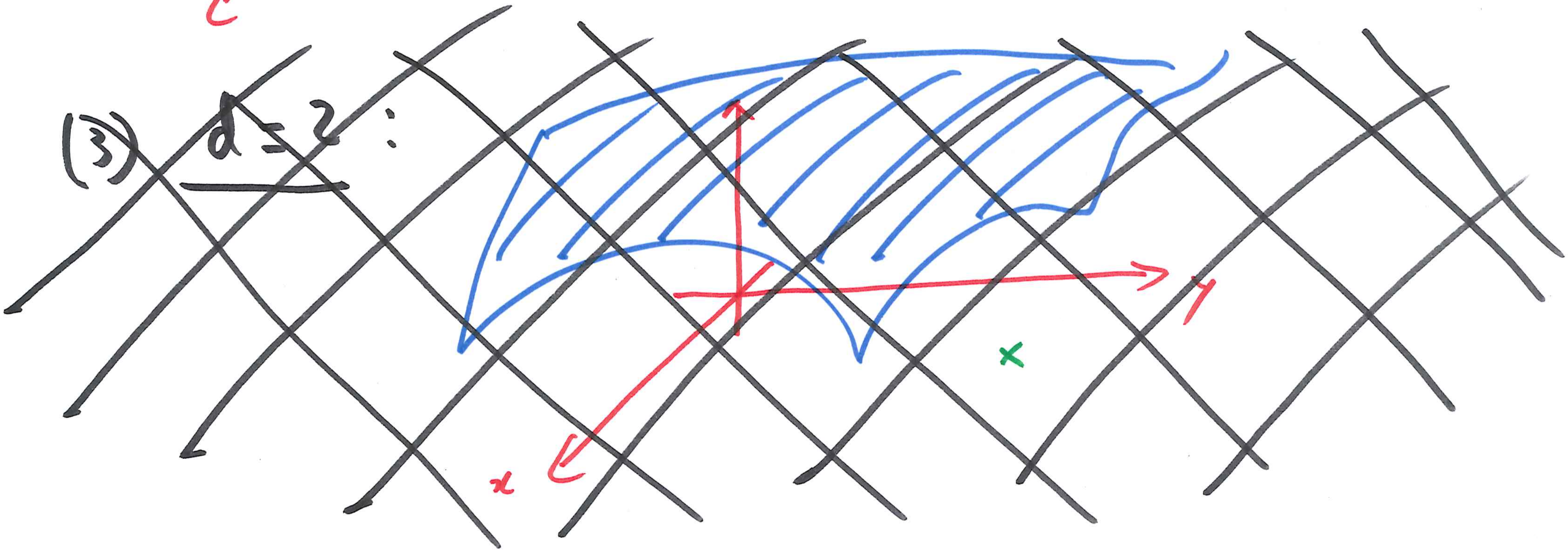
(1) d = 2 : $f(x) = (f_1(x), f_2(x))$



(2) $\underline{d=3}$:

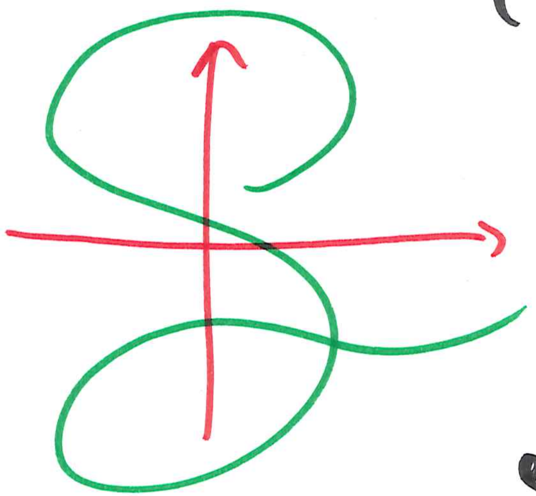


(3) $\underline{d=2}$:



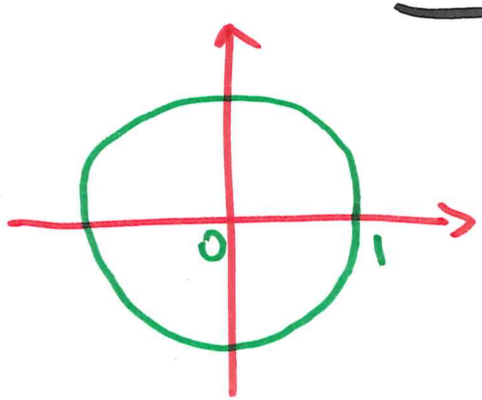
Note: if we represent f by looking at the "green curve"

$$\{ (f_1(x), f_2(x)) \mid x \in I \}$$



Then different f can give the <u>same</u> curve

ex.



$$f(x) = (\cos(x), \sin(x))$$
$$g(x) = (\cos(x^2), \sin(x^2))$$
$$(x \in \mathbb{R})$$

Chapter 2

Differential equations

Equations where:

(1) unknown is a <u>function</u> f

(2) the condition on f relates at any x the values of $f(x)$ and of $f'(x), \ldots, f^{(k)}(x)$.

Ex.

(1) $f' = f$

(a solution is $f(x) = e^x$; not unique: ae^x is another solution for all $a \in \mathbb{R}$)

(2) $f'(x+1) = f(x)$: <u>not</u> an $\underbrace{\text{"ordinary" diff. equ.}}_{\text{ODE}}$

(3) $e^{x+1} f'(x)^2 = x f(x)^3 - f(x) + 1$

ex. $a(t) = e^{t^2}$

(4) $f' = a$, a <u>fixed function</u> of x

Fundamental Th of Calculus (Analysis I) $\Rightarrow$ if a is continuous there is a solution

$$f(x) = \int_{x_0}^{x} a(t)\,dt$$

(5) (Almost) All laws of physics are based on some kind of diff. equations

Newtonian mechanics:

a particle with mass $m>0$

subject to forces (gravity, ...)

has trajectory

$$\gamma(t) = (x(t), y(t), z(t))$$

defined by the ODE:

$$m\gamma''(t) = \text{sum of forces on the particle at time } t$$

$$= \text{expression in terms of } \gamma(t) \text{ and } \gamma'(t)$$

Most important functions satisfy simple ODEs:

$$f(x) = e^{x} \longrightarrow f' = f$$

$$f(x) = \cos(x) \text{ or } = \sin(x) \longrightarrow f'' = -f$$

$$f(x) = e^{-x^2} \longrightarrow f' = -2x f$$

$$\vdots$$

<u>Notation:</u>

y instead of $f(x)$

y' —— $f'(x)$

$\vdots$

$y^{(3)}$ —— $f^{(3)}(x)$

x is used for variable

<u>ex.</u> $y' = -2xy$ ($f(x) = e^{-x^2}$) is a solution

<u>Initial conditions</u> are used to have unique solutions; they specify

the value of $f, f', \dots$ at an <u>initial</u> value of x

<u>Rule</u>: to have uniqueness, specify $f(x_0), \dots, f^{(k-1)}(x_0)$ where the ODE involves derivatives up to (and including) the k-th-derivative.

<u>General Th. (2.1.6)</u>: an ODE has always a solution, at least near the starting time.

2.2. Linear ODEs

Def. A linear ODE (of order k) has the form

$$(E) \qquad y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_1 y' + a_0 y = b$$

where $b, a_0, \dots, a_{k-1}$ are continuous functions from I to $\mathbb{R}$ (or $\mathbb{C}$).

If $b = 0$, we say that the equation is homogeneous, otherwise inhomogeneous.

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Solving (E) means finding all $f : I \longrightarrow \mathbb{C}$, k-times differentiable, s.t. for all $x \in I$

$$f^{(k)}(x) + a_{k-1}(x) f^{(k-1)}(x) + \dots + a_1(x) f'(x) + a_0(x) f(x) = b(x)$$

An initial condition for (E) is specifying:

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(k-1)}(x_0) &= y_{k-1} \end{aligned}\right\} k \text{ conditions}$$

for some $x_0 \in I$

(for some $y_i \in \mathbb{C}$)

Ex.

(1) $$\left.\begin{array}{l} y' = y \Leftrightarrow y' - y = 0 \\ y'' = -y \Leftrightarrow y'' + y = 0 \\ y' = -2xy \end{array}\right\}$$ all linear + homogeneous

(2) $(y')^2 = y^3 - y - 1$: not linear

(3) $y'' + \underbrace{\cos(x)}_{a_0(x)} y = \underbrace{x^2 + 1}_{b(x)}$: linear, inhomogeneous

(4) $y' y + \sin(x) y'' = 1$: not linear

<u>Theorem</u> (2.2.3) Consider (E):

$$y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_1 y' + a_0 y = b$$

(1) The set S_0 of solutions <u>when $b=0$</u> form a vector space of dimension k (over $\mathbb{C}$, or over $\mathbb{R}$ if all coeff. a_i are real-valued).

(2) For any initial condition there is a <u>unique</u> solution.

(3) For b arbitrary, the set S_b of solutions is:

$$S_b = \{ f + f_0 : f \in S_0 \}$$ where

f_0 is <u>one</u> "particular" solution of (E)

<u>Remark:</u> similar to solutions of linear equations

$$A \cdot x = b$$

matrix

vectors

<u>Ex.</u> Why is (3) true?

$$y'' + 2x^2 y' + xy = b$$

<u>Suppose</u> f_0 is a solution; then y is a solution means

(19)

$$y'' + 2x^2 y' + xy = f_0'' + 2x^2 f_0' + x f_0$$

$$\iff \underbrace{(y-f_0)''}_{g''} + 2x^2 \underbrace{(y-f_0)'}_{g'} + x \underbrace{(y-f_0)}_{g} = 0$$

$$\iff y - f_0 = g \in S_0$$

$$\iff y = \underbrace{(y-f_0)}_{\in S_0} + f_0$$