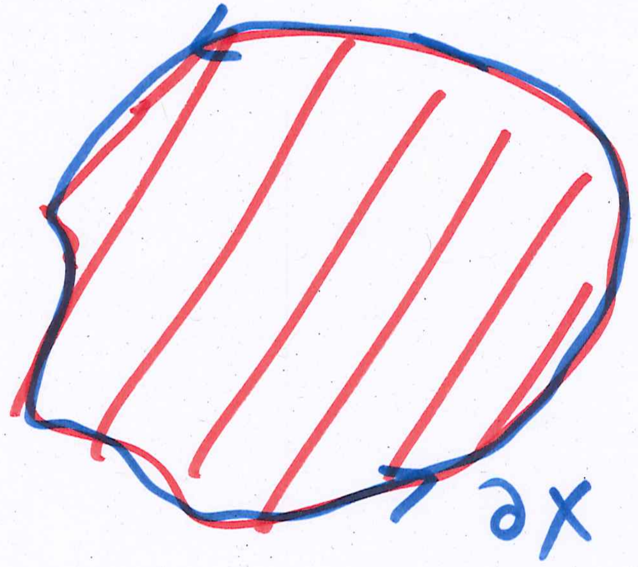


$$\mathrm{Area}(X) = \int_{\partial X} (0, x) \cdot d\vec{s} = \int_a^b \gamma_1(t)\gamma_2'(t)\,dt$$

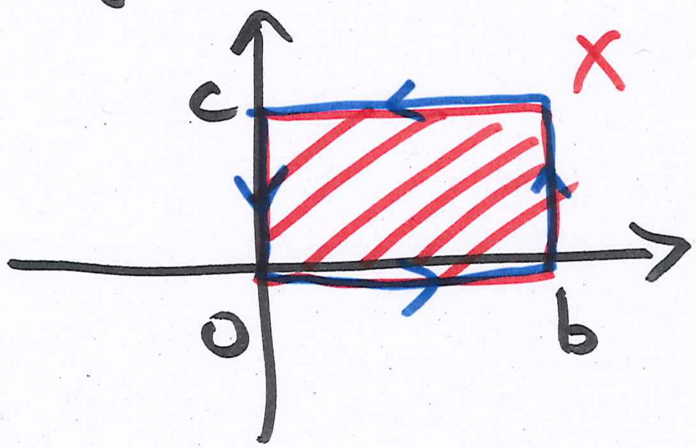
if $\gamma : [a, b] \longrightarrow \mathbb{R}^2$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t))$$

This can be used to compute concretely the areas of various shapes.

Examples: The formula is useful in both directions:

(1) To compute a line integral:



$X = [0,b] \times [0,c]$

The ~~[illegible]~~ Green formula applies, despite corners.

For instance

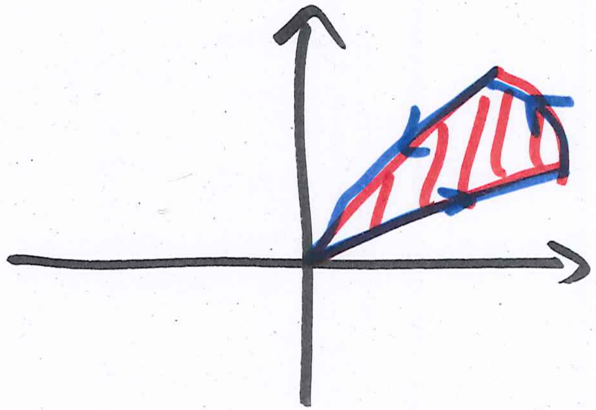
$$f(x,y) = (xy,\ x^2 - y^2)$$

and $b = c = 1$

$$\int_{\partial X} f \cdot d\vec{s} = \int_0^1 \int_0^1 (2x - x)\, dx\, dy$$

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

(Also useful for a disc, using polar coordinates, etc ...)



(2) To compute <u>any</u> 2-dim. integral:

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \int_{\partial X} f \cdot d\vec{s}$$

To compute

symbol

$$\int_X g(x,y)\,dy\,dx = \int_X g(x,y)\,dx\,dy = \,?$$

We need to find a vector field $f = (f_1, f_2)$ s.t. $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = g$.

One can take $f_1 = 0$ and f_2 is obtained as a primitive of g with respect to the first variable.

<u>Ex.</u> $g(x,y) = e^{xy}$

We can take $\begin{cases} f_1 = 0 \\ f_2 = \frac{1}{y} e^{xy} \end{cases}$

Variant of the Green formula

Want to compute

$$\int_X \operatorname{div}(f)\, dx\, dy \quad \operatorname{div}(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

for $f = (f_1, f_2)$ on $\mathbb{R}^2$, where:

Def. $f = (f_1, \dots, f_n)$ vector field on $\mathbb{R}^n$, C^1, then the divergence of f is the function on $\mathbb{R}^n$ defined by

$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n}$$

Define $\tilde{f}(x,y) = (-f_2(x,y), f_1(x,y))$ 297

Then $\tilde{f}$ is C^1 on $\mathbb{R}^2$ and

$$\frac{\partial \tilde{f}_2}{\partial x} - \frac{\partial \tilde{f}_1}{\partial y} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

$$= \operatorname{div}(f)$$

so by the original Green formula we get

$$\int_X \operatorname{div}(f)\, dx\, dy = \int_{\partial X} \tilde{f} \cdot d\vec{s}$$

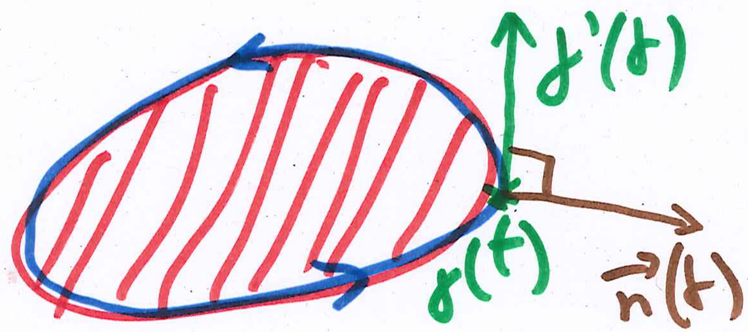
Note that this line integral is

$$\int_a^b \left(-f_2(\gamma(t)) \cdot \gamma_1'(t) + f_1(\gamma(t)) \cdot \gamma_2'(t) \right) dt$$

[for $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ giving ∂X]

$$= \int_a^b f(\gamma(t)) \cdot \vec{n}(t)\, dt$$

where $\vec{n}(t) = (\gamma_2'(t), -\gamma_1'(t))$.



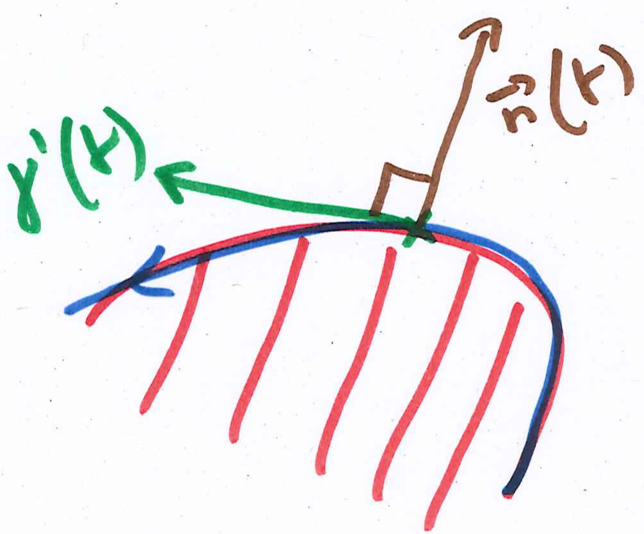
Note: $\gamma'(t) \cdot \vec{n}(t) = 0$

$\vec{n}(t)$ is characterized by:

$$\|\vec{n}(t)\| = \|\gamma'(t)\|$$

$$\vec{n}(t) \perp \gamma'(t)$$

$\vec{n}(t)$ goes "towards the outside" of X



$\vec{n}$ is called the "exterior normal vector"

<u>So</u>: we write the <u>divergence form</u> of the Green formula

$$\int_X \operatorname{div}(f)\, dx\, dy = \underbrace{\int_{\partial X} f \cdot d\vec{n}}_{\text{independent of reparameterization, like line integrals}}$$

<u>Ex.</u> Want to compute

$$\int_X \Delta g \; dx\, dy \qquad (C^2)$$

where $g : \mathbb{R}^2 \to \mathbb{R}$

and $\Delta g = \dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2}$ (Laplacian)

This can be obtained from the divergence Green formula for the vector field $f = \nabla g$: indeed

$$\mathrm{div}(\nabla g) = \frac{\partial (\nabla g)_1}{\partial x} + \frac{\partial (\nabla g)_2}{\partial y}$$

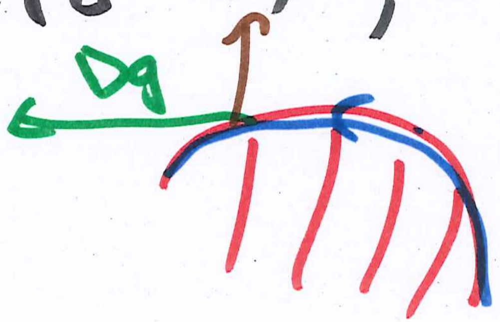
$$= \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \Delta g$$

since $\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right)$

So

$$\boxed{\int_X \Delta g(x,y)\,dx\,dy = \int_{\partial X} \nabla g \cdot d\vec{n}}$$

<u>Eg:</u> if, on ∂X, ∇g is tangent to ∂X, $\nabla g \perp \vec{n}$, so



$$\boxed{\int_X \Delta g\,dx\,dy = 0.}$$