$$\gamma_1 : [a,b] \longrightarrow \mathbb{R}^n$$

$$\gamma_2 = \gamma_1 \circ \sigma \quad , \quad \sigma : [\overset{u}{c}, d] \longrightarrow [\overset{t}{a}, b]$$

$$\int_{\gamma_2} f(s) \cdot d\vec{s} \underset{\text{def}}{=\!=} \int_c^d f(\underbrace{\gamma_1(\sigma(u))}_{\gamma_2(u)}) \cdot \underbrace{\gamma_2'(u)}_{\substack{\| \text{ chain rule} \\ \sigma'(u)\, \gamma_1'(\sigma(u))}} \, du$$

$$= \int_c^d \underbrace{f(\gamma_1(\sigma(u)))}_{\text{vector}} \cdot \underbrace{\sigma'(u)}_{\text{number}} \underbrace{\gamma_1'(\sigma(u))}_{\text{vector}} \, du$$

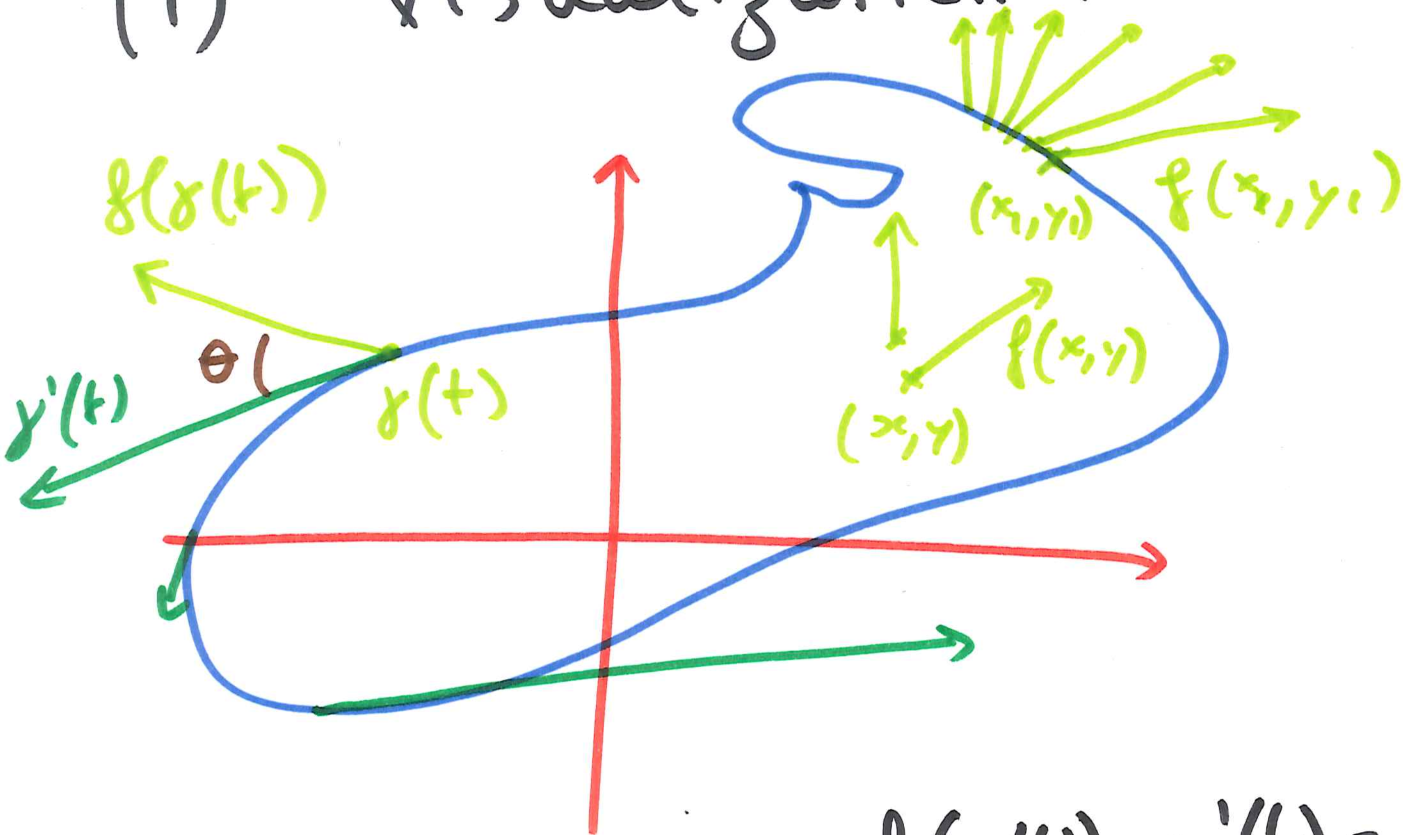
<u>Change of variable</u> (Analysis I): $t = \sigma(u)$

$dt = \sigma'(u)\, du$

$$= \int_a^b f(\gamma_1(t)) \cdot \gamma_1'(t) \, dt \underset{\text{def}}{=} \int_{\gamma_1} f(s) \cdot d\vec{s}$$

Remarks:

(1) Visualization:



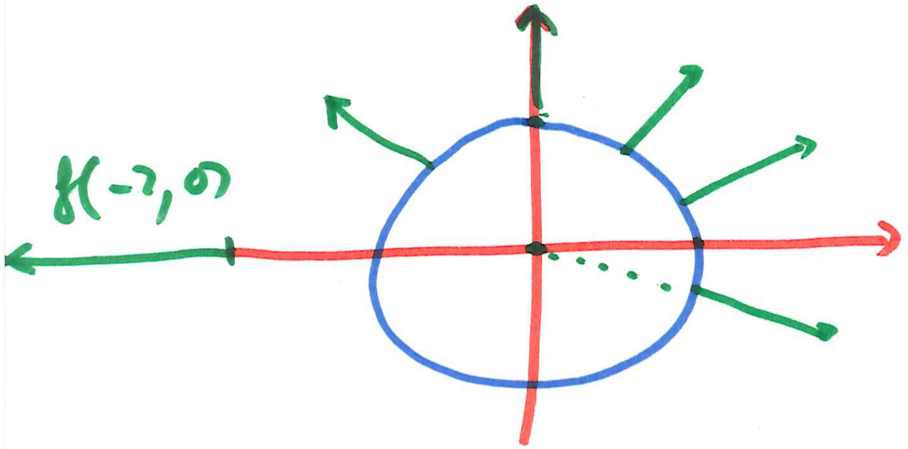
$\gamma : [a,b] \longrightarrow \mathbb{R}^2$

curve (with speed indications)

$f : X \longrightarrow \mathbb{R}^n$

"vector field"*

$$f(\gamma(t)) \cdot \gamma'(t) = \|\gamma'(t)\| \cdot \|f(\gamma(t))\| \cdot \cos(\theta)$$



$\gamma(t) = (\cos t, \sin t)$

$0 \leq t \leq 2\pi$

$f(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \quad f(s) \cdot d\vec{s} = 0$ on the curve

so $\int_\gamma f(s) \cdot d\vec{s} = 0$

(2) Motivation / physical interpretations?

relation with 2-dim. integrals (later)

Newtonian mechanics:

γ: trajectory of a particle (t = time)

f: vector field of all forces exerted on particle

$$\Longrightarrow \int_\gamma f(s) \cdot d\vec{s} = \text{"work of the forces along the trajectory"}$$

$=$ difference between kinetic energy at $\gamma(b)$ and at $\gamma(a)$

Conservative vector fields

Ex. $X \subset \mathbb{R}^n$ open

Let $f = \nabla g$, $g : X \longrightarrow \mathbb{R}$ differentiable

$$f(x) = \begin{pmatrix} \partial_{x_1} g \\ \vdots \\ \partial_{x_n} g \end{pmatrix} \quad (\text{"gradient field"})$$

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a curve with image contained in X.

Fact: $$\boxed{\int_\gamma f(s) \cdot d\vec{s} = g(\gamma(b)) - g(\gamma(a))}$$

Why? $\boxed{\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))}$ 197

$$\int_\gamma f(s) \cdot d\vec{s} = \int^b f(\gamma(t)) \cdot \gamma'(t) \, dt$$

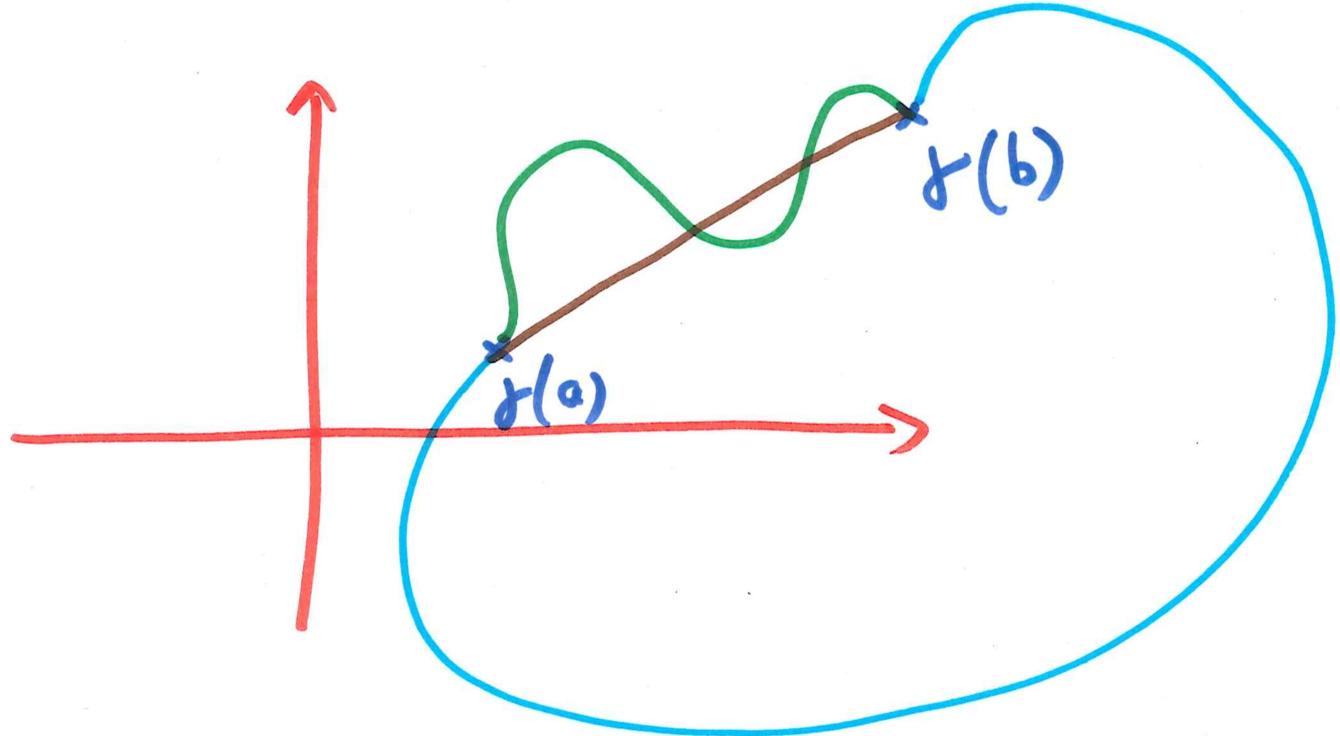
$$= \sum_{i=1}^{n} \int_a^b (\partial_{x_i} g)(\gamma(t))\, \gamma_i'(t) \, dt$$

$$= \cancel{\sum_{i=1}^{n}} \int_a^b \underbrace{\sum_{i=1}^{n} (\partial_{x_i} g)(\gamma(t))\, \gamma_i'(t)}_{\text{derivative of } g \circ \gamma : [a,b] \to \mathbb{R}} \, dt$$

(chain rule)

$$= \int_a^b (g \circ \gamma)'(t) \, dt$$

fundamental th. of calculus (Analysis I) $\longrightarrow$ $= \quad g \circ \gamma(b) - g \circ \gamma(a)$



$f = \nabla g$

$$\int f \cdot d\vec{s} = \int f \cdot d\vec{s} = \int f \cdot d\vec{s}$$

The integral only depends on the end-points of the curve!

Def. (1) A <u>vector field</u> f is <u>conservative</u> if the line integrals of f only depend on the extremities of the curves involved.

(2) If $f = \nabla g$ then g is called a <u>potential</u> for f

[It is not unique, for instance $g+$ (constant) is also a potential]

Facts:

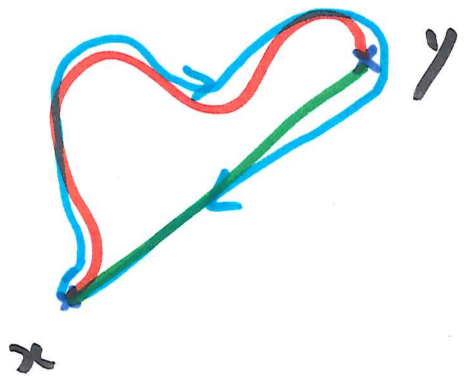
(1) (4.1.10) Any continuous conservative vector field f (on an open set $X \subset \mathbb{R}^n$) is a gradient field $f = \nabla g$ for some potential $g : X \longrightarrow \mathbb{R}$ (differentiable).

(2) "One can compute a potential g, given f, by integration with respect to successive variables".

(3) A vector field f is conservative $\iff$ for any closed curve $\gamma : [a,b] \longrightarrow X$, we have

$$\int_\gamma f(s) \cdot d\vec{s} = 0$$

$\gamma(a) = \gamma(b)$



γ_1

γ_2

γ_3 : closed

__Ex.__ (How to find a potential for f?)

$$f(x,y) = \begin{pmatrix} y^2\cos(xy^2) - 1 \\ 2xy\cos(xy^2) \end{pmatrix}$$

__Idea:__ integrate with respect to x, then to y.

__Goal:__ find $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t.

$$\partial_x g = y^2\cos(xy^2) - 1$$
$$\partial_y g = 2yx\cos(xy^2)$$

?

w.r.t x: $\quad g = \sin(xy^2) - x + h(y)$

Then $\quad \partial_y g = 2xy\cos(xy^2) + h'(y)$

$\qquad\qquad \stackrel{?}{=} 2xy\cos(xy^2)$

so $h'(y) = 0$ so h is a constant

so

$$\boxed{g(x,y) = \sin(xy^2) - x} \quad (+\text{ constant})$$

Criterion for a vector field to be conservative

Observation: if $f: X \longrightarrow \mathbb{R}^n$ is conservative and $f = (f_1, \dots, f_n)$

Then $\quad f_i = \partial_{x_i} g \quad , \quad 1 \leq i \leq n$

so **if** g is of class C^2 then

$$\boxed{\partial_{x_j} f_i} = \partial_{x_j} \partial_{x_i} g = \partial_{x_i} \partial_{x_j} g = \boxed{\partial_{x_i} f_j}$$

$1 \leq i < j \leq n$

This is a <u>necessary condition</u> for f to be conservative, <u>if f is C^1</u>.

<u>Ex:</u>

$$f(x,y) = \begin{pmatrix} y^2 \cos(xy^2) - 1 \\ 2y \cos(xy^2) \end{pmatrix}$$

If f is conservative then

$$\partial_y \left(y^2 \cos(xy^2) - 1 \right) \overset{?}{=} \partial_x \left(2y \cos(xy^2) \right)$$

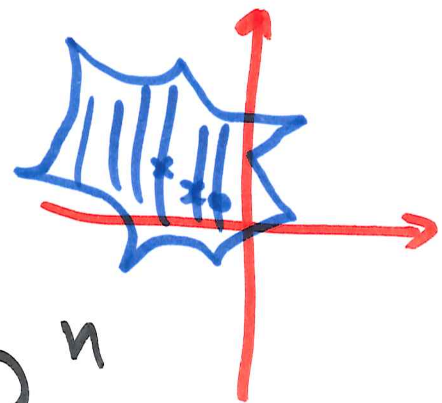
$$2y \cos(xy^2) - 2xy^3 \sin(xy^2) \overset{?}{=} -2y^3 \sin(xy^3)$$

Not the case, for instance when $x = 0$.

<u>Question:</u> is this condition also <u>sufficient</u> for f to be conservative ?

This depends on the <u>shape</u> of the open set X where f is defined !

<u>Th.</u> (4.1.17) Suppose that $X \subset \mathbb{R}^n$ is <u>open</u> and "star-shaped" (w.r.t $x_0 \in X$)



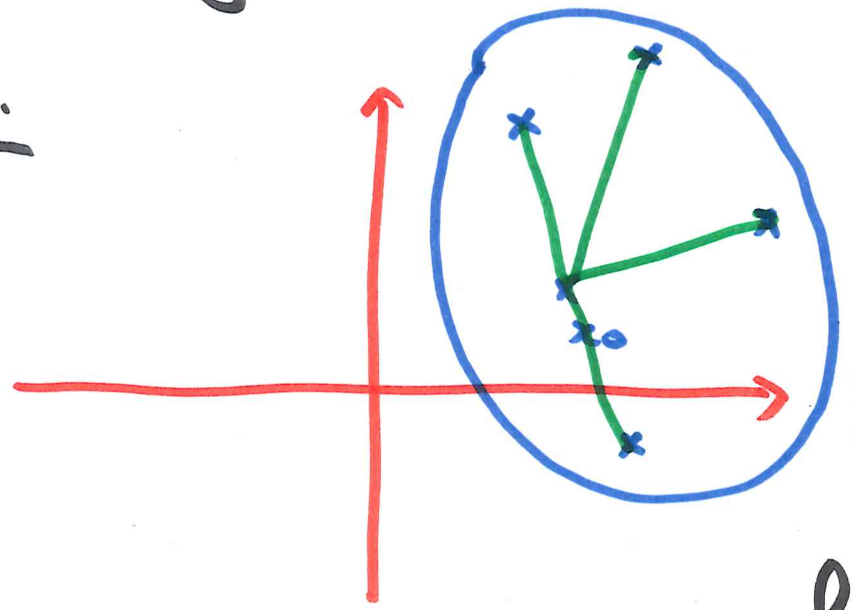
<u>Then</u> any C^1 vector field $f: X \longrightarrow \mathbb{R}^n$ s.t.

$$\partial_{x_i} f_j = \partial_{x_j} f_i, \quad 1 \leq i, j \leq n$$

is <u>conservative.</u>

<u>Def.</u> $X \subset \mathbb{R}^n$, $x_0 \in X$, is star-shaped if X contains the line-segment joining x_0 to x.

<u>Ex.</u>

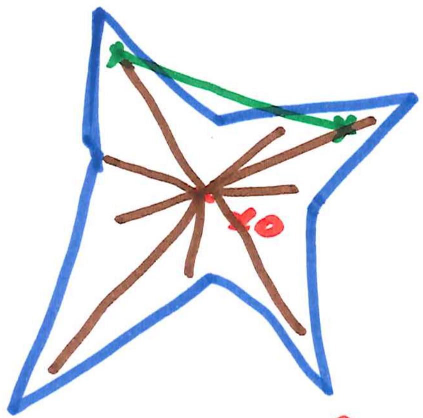


X <u>convex</u> (for any x, y in X, the line segment from x to y is in X.)

For Instance:

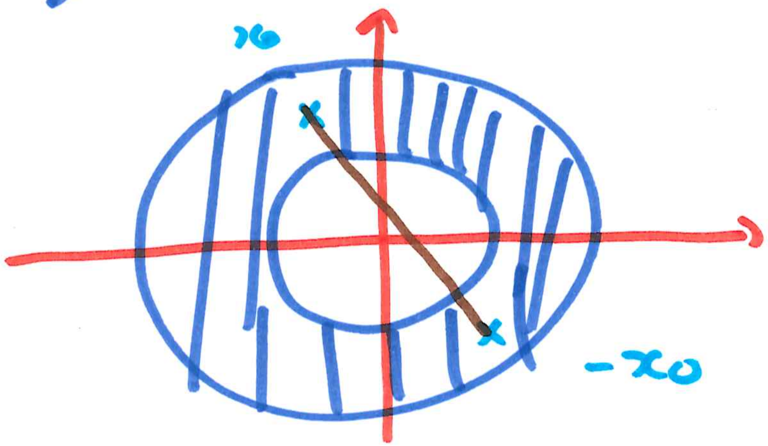
(convex) $\begin{cases} \mathbb{R}^n \\ \text{Open ball} \\ [a_1, b_1] \times \dots \times [a_n, b_n] \end{cases}$

Ex.



X is star-shaped but <u>not</u> convex

Ex.

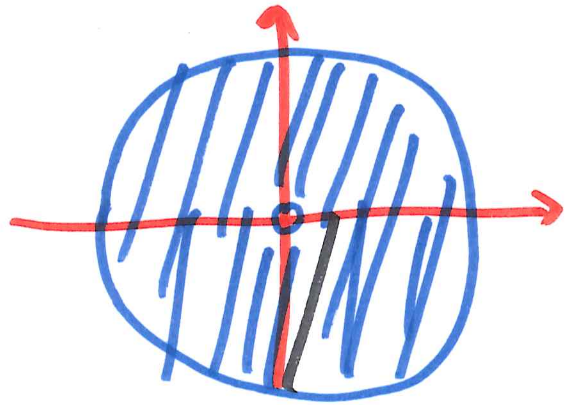


$X = \{(x, y) \mid 1 < x^2 + y^2 < 2\}$

<u>not</u> star-shaped

Important: there exist $X \subset \mathbb{R}^n$ open, for which Th. 4.1.17 is **not** true (the necessary condition is not enough for a vector field to be conservative).

Ex. (4.1.19) $X = \{(x,y) \mid 0 < x^2+y^2 < 2\}$



$$f(x,y) = \begin{pmatrix} -\dfrac{y}{x^2+y^2} \\[2ex] \dfrac{x}{x^2+y^2} \end{pmatrix} \Bigg| \; C^1 \text{ on } X$$

1 f satisfies the necessary condition:

$$\partial_y\left(\frac{-y}{x^2+y^2}\right) \stackrel{?}{=} \partial_x\left(\frac{x}{x^2+y^2}\right)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2} \stackrel{?}{=} \frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}$$

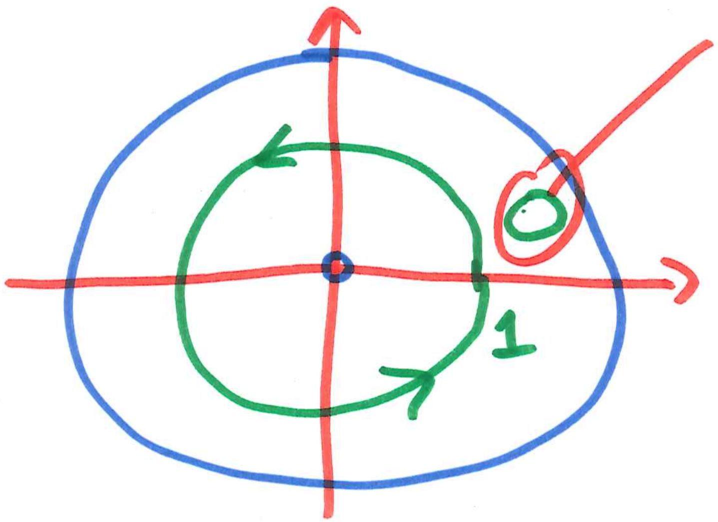
$$\|$$

$$\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} \stackrel{?}{=} \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

2) f is not conservative:

There is a closed curve γ in X s.t.

$$\int_\gamma f(s) \cdot d\vec{s} \neq 0$$



such a curve would not work!

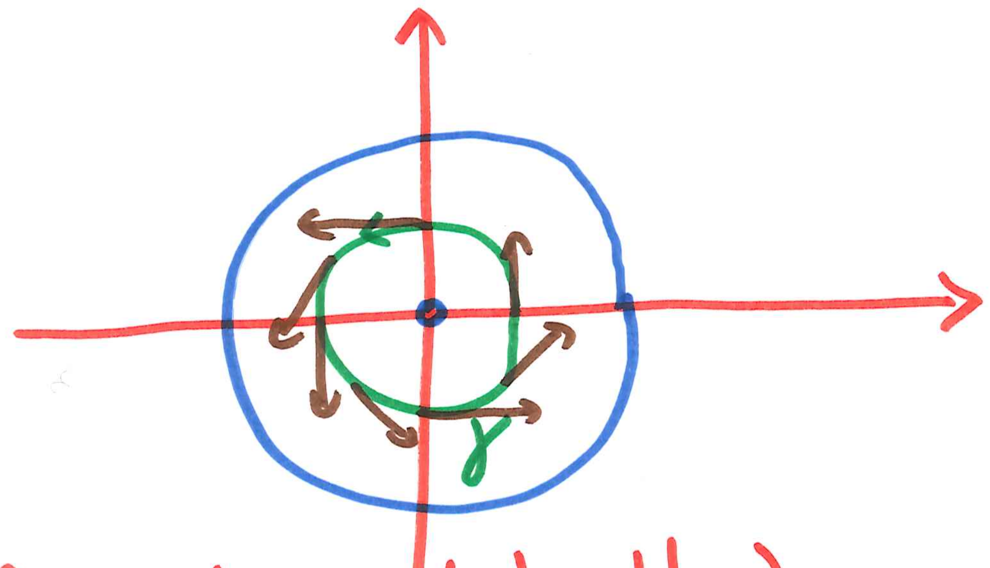
$$\gamma(t) = (\cos t, \sin t) \quad (0 \le t \le 2\pi)$$

$$f(x,y) = \begin{pmatrix} \dfrac{-y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{pmatrix}$$

so $f(\gamma(t)) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

In fact,

$$f(\gamma(t)) = \gamma'(t)$$



$$\int_\gamma f(s)\cdot d\vec{s} = \int_0^{2\pi} \|\gamma'(t)\|^2 \, dt = 2\pi \neq 0$$

(f is tangent to the curve)