2.3 - Linear ODE of order 1

$$y' + a y = b$$

functions

Step 1 - (Homogeneous equation)

$$y' + ay = 0$$

$$\Rightarrow \log(|y|)' = \frac{y'}{y} = -a$$

$$\log(|y|)' = -a$$

$$\Rightarrow \quad \log(|y|) = \underbrace{-\int_{x_0}^{x} a(t)\,dt}_{-A(x)\ (\text{primitive of } a)} + c$$

for some $c \in \mathbb{C}$

$$\Rightarrow \quad \boxed{y = z\exp(-A(x))}$$

for some $z \in \mathbb{C}$

<u>Now</u> we check that this formula gives <u>all</u> solutions to

$$y' + ay = 0.$$

Let $z \in \mathbb{C}$

and $f(x) = z \exp(-A(x))$, $A' = a$

Then (chain rule): for every $x \in I$

$$f'(x) = z\,(-A')(x) \exp(-A(x))$$
$$= -z\,a(x) \exp(-A(x)) = -a(x) f(x)$$
$$\Rightarrow \quad f' + af = 0.$$

To check that we get **all** solutions of $y' + ay = 0$ this way:

take a solution f

Q.: is $f(x) = z \exp(-A(x))$?

$$\Updownarrow$$

$$f(x)\exp(A(x)) = z \;?$$

$$\Updownarrow$$

$$\left(f(x)\exp(A(x))\right)' = 0 \;?$$

$$\Updownarrow$$

$$f'(x)\exp(A(x)) + f(x)a(x)\exp(A(x)) = 0 \;?$$

$$-a(x)f(x)\exp(A(x)) + f(x)a(x)\exp(A(x)) \overset{?}{=} 0$$

ok!

<u>In practice</u>: use the formal argument (or try to <u>remember</u> the formula).

With initial condition:

$$f(x) = z\exp(-A(x))$$

Want: $f(x_0) = y_0$; then need

$$z\exp(-A(x_0)) = y_0$$

so $z = y_0 \exp(A(x_0))$

$\Rightarrow f(x) = y_0 \exp(A(x_0) - A(x))$

<u>Step 2 -</u> (Inhomogeneous equation)

$$y' + ay = b$$

Need to find <u>one</u> particular solution f_0 ; then the solutions are

$$f(x) = f_0(x) + z \exp(-A(x)), \quad z \in \mathbb{C}$$

$$y' + ay = b$$

Method 1: make a guess for f_0

(e.g. a polynomial, b also; then maybe f a polynomial is a good choice)

Method 2: systematic method

"variation of constants"

We look for a solution of the form:

$$f_0(x) = \underbrace{z(x)}_{\text{now a function of } x\text{, not a constant}} \exp(-A(x))$$

$$f_0'(x) = z'(x)\exp(-A(x)) + z(x)(-a(x))\exp(-A(x))$$

$$= \exp(-A(x))\left(z'(x) - a(x)z(x)\right)$$

We want: $f_0'(x) + a(x)f_0(x) = b(x)$

$$\exp(-A(x))\left(z'(x) \cancel{- a(x)z(x)}\right) \cancel{+ a(x)z(x)\exp(-A(x))} = b(x)$$

$\Longleftrightarrow$

$$z'(x) = b(x)\exp(A(x))$$

so we can take z to be any primitive of $b\exp(A)$, so for instance

$$\boxed{f_0(x) = \left(\int_{x_0}^{x} b(t)\exp(A(t))\,dt\right)\exp(-A(x))}$$

Note: often not explicitly computable!

Another trick:

$$y' + ay = b_1 + b_2$$

$$(x + \cos(2x))$$

if you know a solution f_1 of

$$y' + ay = b_1$$

and f_2 of $y' + ay = b_2$, then

$f_0 = f_1 + f_2$ solves $y' + ay = b$:

$$f_0' + af_0 = (f_1' + af_1) + (f_2' + af_2) = b_1 + b_2$$

2.4 - Linear ODEs with <u>constant</u> coefficients

$k \geq 1$

(E) $\quad y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_1 y' + a_0 y = b$

where $a_i \in \mathbb{C}$ (constants)

(<u>but</u> b can be a function!)

<u>Ex.</u> $y'' + y = 0$, $\quad y'' + y' - y = \cos(x)$ — both constant coeff.

$y' + xy = 0$ — <u>not</u> constant coeff.

Step 1 _ (Homogeneous equation)

Note: $f(x) = e^{ax}$ solves $f' = af$
$(a \in \mathbb{C})$

$$f'' = a^2 f \ \dots$$
$$\vdots$$
$$f^{(j)} = a^j f$$

so for $y = f$ we get

$$y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_1 y' + a_0 y =$$
$$\left(a^k + a_{k-1} a^{k-1} + \dots + a_1 a + a_0\right) e^{ax}$$

If a is a solution of the polynomial equation

(32) $X^k + a_{k-1} X^{k-1} + \cdots + a_1 X + a_0 = 0$

Then $f(x) = e^{ax}$ is a solution of the ODE.

$P(X)$ "characteristic polynomial of the ODE"

Recall: "fundamental theorem of algebra"

$P(X) = (X - \alpha_1) \cdot \ldots \cdot (X - \alpha_k)$, for $\alpha_i \in \mathbb{C}$

so $f_i(x) = e^{\alpha_i x}$ for $1 \leq i \leq k$

are solutions of the (homog.) ODE.

Any sum $z_1 f_1 + \cdots + z_k f_k$ is also sol. $(z_i \in \mathbb{C})$

Did we get all solutions?

It depends on P!

<u>Case 1</u> – all α_i are distinct (P has k distinct roots in $\mathbb{C}$)

Then the functions $f_1, \dots, f_k$ form a <u>basis</u> of the space of (complex) solutions of

$$y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_1 y' + a_0 y = 0$$

Case 2 – some α_i coincide

(ex. $P(X) = (X-1)^2$: $\alpha_1 = \alpha_2 = 1$)

To find a basis of solutions in such a case: – for each root α_i that appears $d \geq 2$ times [i.e. $P(X) = (X-\alpha_i)^d \underbrace{(X-\alpha_j)\cdots}_{\alpha_i \text{ does not appear}}$]

– take functions

$$f_{i,0}(x) = e^{\alpha_i x}, \quad f_{i,1}(x) = x e^{\alpha_i x}, \quad \ldots$$

$$f_{i,d-1}(x) = x^{d-1} e^{\alpha_i x}$$

– take all these k functions: they form a basis

35

Ex. $P(X) = (X-1)^2 (X+i)(X-i)X^3$

$\hookrightarrow \alpha_5 = \alpha_6 = \alpha_7 = 0$

Basis:

(real-valued basis)

e^x	$\alpha_1 = \alpha_2 = 1$	e^x
xe^x		xe^x
e^{-ix}	$\alpha_3 = -i$	$\cos(x)$
e^{ix}	$\alpha_4 = i$	$\sin(x)$
$1 = e^{0 \cdot x}$	$\alpha_5 = \alpha_6 = \alpha_7 = 0$	1
x		x
x^2		x^2

A solution is of the form

$$\boxed{z_1 e^x + z_2 x e^x + z_3 \underbrace{e^{-ix}}_{\cos(x)} + z_4 \underbrace{e^{ix}}_{\sin(x)} + z_5 + z_6 x + z_7 x^2}$$

$(z_i \in \mathbb{R})$

What about real-valued solutions?

Even if $a_i \in \mathbb{R}$, the roots α_i of $P(x)$ might be complex numbers (not in $\mathbb{R}$).

Ex. $x^2+1=0 \iff x = i$ or $-i$

Sol. e^{ix}, e^{-ix}

We construct real-valued solutions by taking suitable combinations:

$\cancel{e^{\alpha x}, \; e^{\bar{\alpha} x} \longrightarrow \cos(\alpha x)}$ $\cancel{(\alpha \in \mathbb{R})}$

(1) $\alpha \in \mathbb{R}$

(2) $\alpha \notin \mathbb{R}$, $\alpha = a + ib$

$\mathbb{C}$-valued basis		$\mathbb{R}$-valued basis
(1) $x^j e^{\alpha x}$	$\rightsquigarrow$	$x^j e^{\alpha x}$
(2) $e^{\alpha x}$, $e^{\bar{\alpha} x}$ $e^{\alpha x} = e^{ax} e^{ibx}$, $e^{\bar{\alpha} x} = e^{ax} e^{-ibx}$	$\rightsquigarrow$	$e^{ax} \cos(bx)$, $e^{ax} \sin(bx)$
$x^j e^{\alpha x}$, $x^j e^{\bar{\alpha} x}$	$\rightsquigarrow$	$x^j e^{ax} \cos(bx)$, $x^j e^{ax} \sin(bx)$

Step 2 - (Inhomogeneous case)

$$y^{(k)} + \cdots + a_1 y' + a_0 y = b$$ → function

Method 1 - (Good guess)

(1) If b is a <u>polynomial</u> of degree d, look for a particular solution $f_0(x)$ which is a polynomial of degree

d (in general, if $a_0 \neq 0$) or

$d+1$ (if $a_0 = 0$, $a_1 \neq 0$) or

$d+2$ ...

Ex: $y^{(k)} = x^d + \cdots \rightsquigarrow y$ of degree $d+k$

(2) If b is of the form

$$\cancel{b(x) = e^{cx}} \quad b(x) = x^d e^{cx}$$

look for a solution

$$f_0(x) = Q(x)\, e^{cx}, \quad \deg(Q) = d$$

<u>if</u> c is not a solution of $P(X) = 0$.

(Otherwise increase $\deg(Q)$).

(3) If b is of the form

$$b(x) = x^d \cos(cx) \quad \left(\text{or } x^d \sin(cx)\right)$$

look for

$$f_0(x) = Q_1(x)\cos(cx) + Q_2(x)\sin(cx)$$

where $\deg Q = d$ (if ic is not a root of P).

Always: if $b = b_1 + b_2 + \dots + b_j$

then a particular solution for ODE with R.H.S b is a sum of f_i, where f_i is a particular solution for b_i

Ex. $b(x) = x^3 + \cos(x) + e^{3x}$