Ex. (of the <u>necessary</u> condition for $f: X \longrightarrow \mathbb{R}^n$, $X \subseteq \mathbb{R}^n$, $f = (f_1, \dots, f_n)$ to be conservative)

<u>$n = 2$:</u> <u>one</u> equation to check: (x, y) variables

$$\partial_x f_2 = \partial_y f_1$$

<u>$n = 3$:</u> three <u>equations</u> : (x, y, z) variables

$$\begin{cases} \partial_x f_2 = \partial_y f_1 \\ \partial_x f_3 = \partial_z f_1 \\ \partial_y f_3 = \partial_z f_2 \end{cases}$$

We encode these three equations by the condition

$$\left(\vec{\mathrm{rot}}(f) =\right) \ \mathrm{curl}(f) = 0$$

where for $\underline{\underline{\text{any}}}$ C^1 vector field

$$f : X \ (\subset \mathbb{R}^3) \longrightarrow \mathbb{R}^3$$

$$\mathrm{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix} : X \longrightarrow \mathbb{R}^3$$

Mnemonic: formal 3×3 determinant

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

((e_1, e_2, e_3) basis vectors in $\mathbb{R}^3$)

$$= e_1 \partial_y f_3 + e_2 \partial_z f_1 + e_3 \partial_x f_2$$
$$- f_1 \partial_y e_3 - f_2 \partial_z e_1 - f_3 \partial_x e_2$$

$$= \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

- <u>$n \geq 4$:</u> more conditions.

4.2 – Riemann integral in $\mathbb{R}^n$

$X \subset \mathbb{R}^n$

$f : X \longrightarrow \mathbb{R}$

Want to define / compute / interpret

. volumes
. areas
. center of mass
. . .

$$\int_X f(x_1, \ldots, x_n)\, dx_1 \ldots \ldots dx_n$$

We use an <u>axiomatic</u> description:

we will describe

- which sets X
- which functions f on X

can be integrated, and properties of the integral which

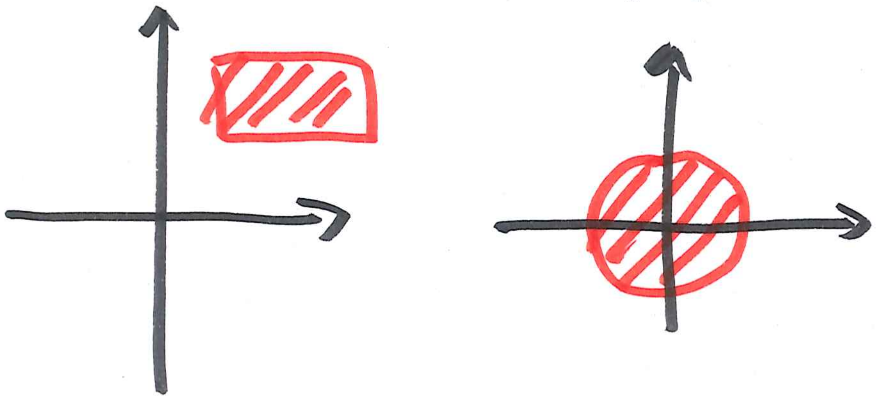
(1) characterize the integrals

(2) allow us to compute the integrals by reduction to dim. 1

Sets: $X \subset \mathbb{R}^n$ is closed and bounded ("compact")

[ex. $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$

or $\{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq R^2\}$]



Functions: $f: X \longrightarrow \mathbb{R}$ continuous

For every X, f as above, there is an integral

$$\int_X f(x)\,dx \in \mathbb{R} \quad \left[\begin{array}{l} x = (x_1, \dots, x_n) \\ dx = dx_1 \dots dx_n \end{array}\right.$$

such that all the following properties hold:

(1) [Compatibility] For $n = 1$, $X = [a, b]$ $(a \le b)$

then $\displaystyle\int_{[a,b]} f(x)\,dx = \int_a^b f(t)\,dt$

Analysis I

Riemann integral

(2) [Linearity] If f_1, f_2 are continuous on X and a_1, a_2 are real numbers, then

$$\int_X \left(a_1 f_1(x) + a_2 f_2(x)\right) dx = a_1 \int_X f_1(x)\, dx + a_2 \int_X f_2(x)\, dx$$

$$\left(\Rightarrow \int_X 0\, dx = 0\right)$$

(3) [Positivity] If $f \geq g$ are continuous on X then $\int_X f(x)\, dx \geq \int_X g(x)\, dx$

(esp. if $f \geq 0$ then $\int_X f(x)\, dx \geq 0$.)

If $Y \subset X$ is also closed and bounded and $f : X \longrightarrow \mathbb{R}$ continuous, and $f \geq 0$ then

$$\int_Y f(x)\,dx \leq \int_X f(x)\,dx$$

(4) [Triangle inequality] We have

$$\left| \int_X f(x)\,dx \right| \leq \int_X |f(x)|\,dx$$

and

$$\left| \int_X (f(x) + g(x))\,dx \right| \leq \int_X |f(x)|\,dx + \int_X |g(x)|\,dx$$

(5) [Fubini formula] $f : X \longrightarrow \mathbb{R}$

$n = n_1 + n_2\ ,\ n_i \geq 1$

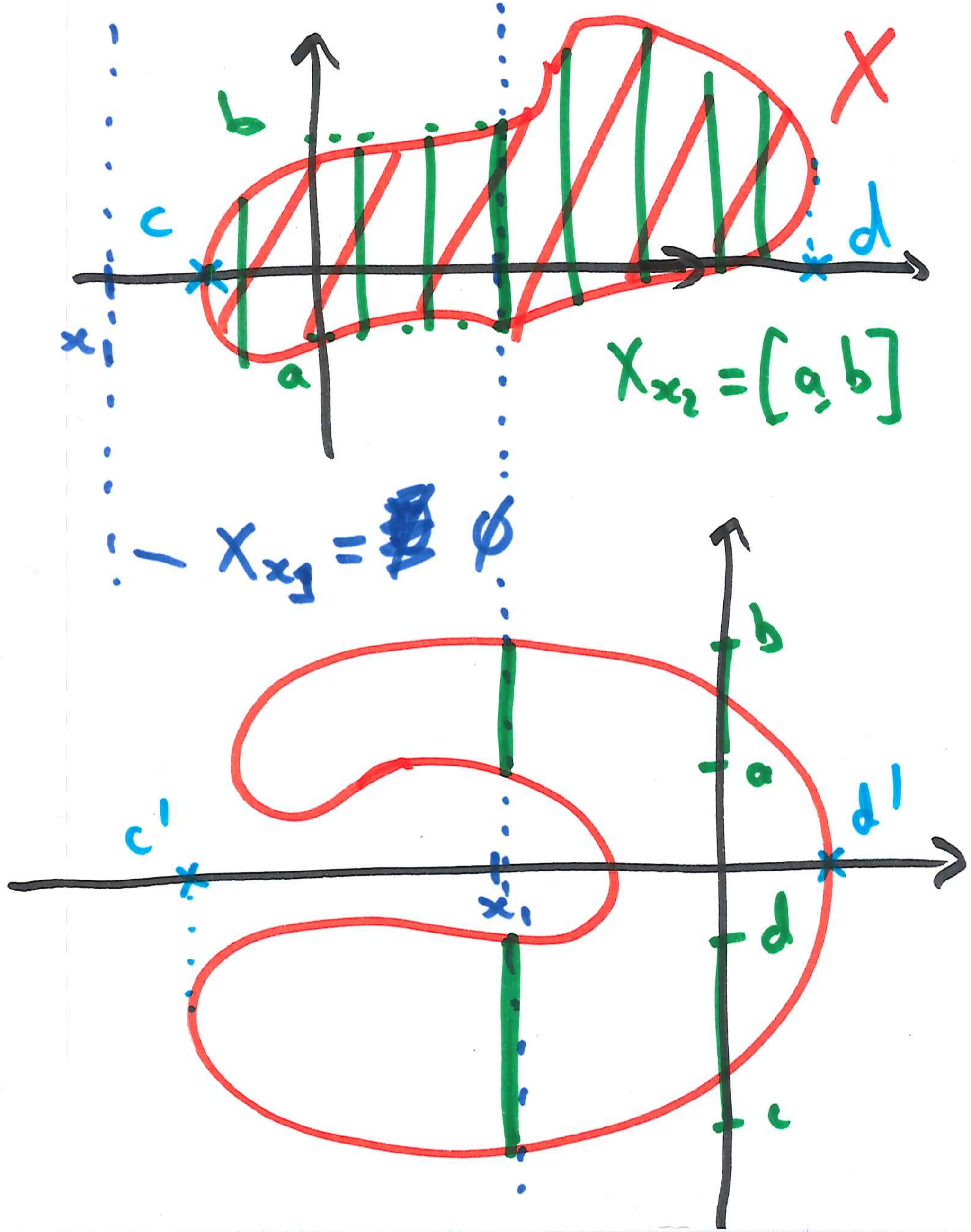
For $x_1 \in \mathbb{R}^{n_1}$, let

$$X_{x_1} = \left\{ \cancel{(x_1, x_2) \in X}\ x_2 \in \mathbb{R}^{n_2} \text{ s.t. } (x_1, x_2) \in X \right\}$$

X_{x_1} is closed and bounded in $\mathbb{R}^{n_2}$

$$\int_X f(x)\,dx = \int_{X_1} \left(\int_{X_{x_1}} f(x_1, x_2)\,dx_2 \right) dx_1$$

$n = 2 = 1 + 1$



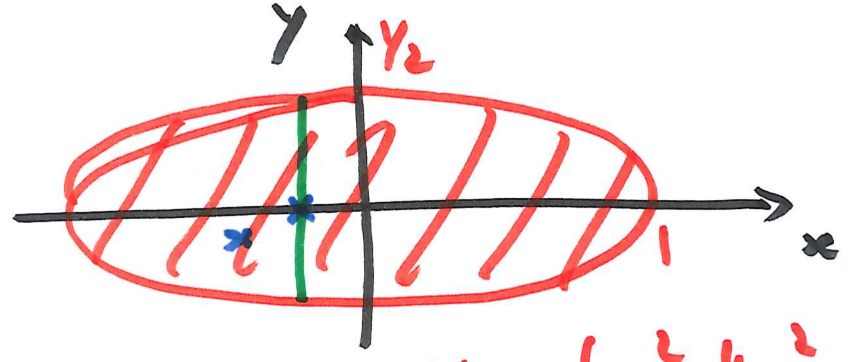
where X_1 is the set of $x_1 \in \mathbb{R}^{n_1}$ s.t. $X_{x_1} \neq \emptyset$ (closed and bounded)

<u>if</u> the function

$$g(x_1) = \int_{X_{x_1}} f(x_1, x_2)\, dx_2$$

is <u>continuous</u>.

$X_1 = [c, d]$



$f(x, y)$

$X_1 = [-1, 1]$

If $-1 \leq x \leq 1$, then

$$X_{x_1} = \left[-\frac{\sqrt{1-x^2}}{2}, \frac{\sqrt{1-x^2}}{2} \right]$$

1-dim. Riemann integral, depending on x

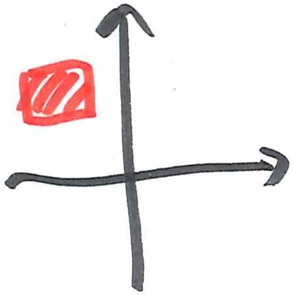
second Riemann integral

$$\int_X f(x,y)\, dx\, dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} f(x,y)\, dy \right) dx$$

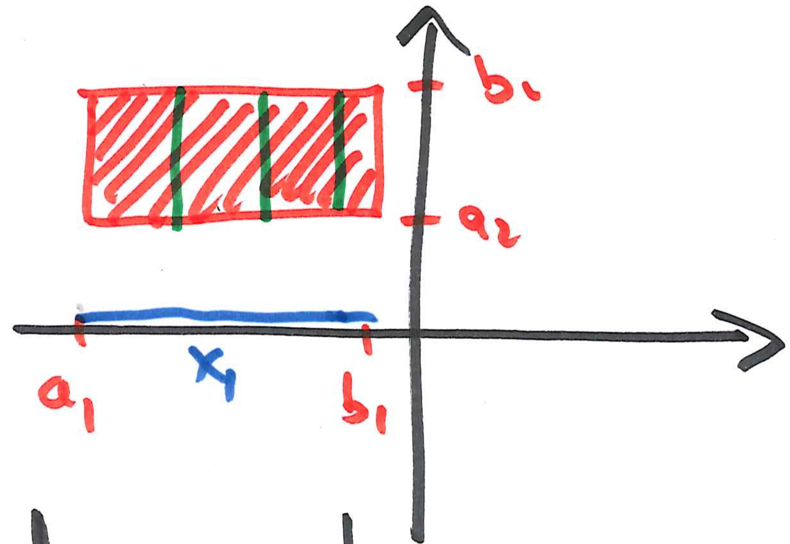
Examples:

(1) $X = [a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$

$f = 1$



$\underline{n = 2} = 1 + 1$



$$\int_X dx_1 dx_2 = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} dy \right) dx$$

$$= \int_{a_1}^{b_1} (b_2 - a_2)\, dx$$

$$= (b_2 - a_2)(b_1 - a_1)$$

$\underline{n=3} = 1+2$

$$X_1 = [a_1, b_1]$$

and $\quad X_{x_1} = \underbrace{[a_2, b_2] \times [a_3, b_3]}_{Y}$ for all x_1

$$\int_X dx_1\, dx_2\, dx_3 = \int_{a_1}^{b_1} \left(\int_Y dx_2\, dx_3 \right) dx_1$$

$$= (b_3 - a_3)(b_2 - a_2)$$

$$= (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

and so on . . . .

$$\int_X dx_1 \ldots dx_n = (b_n - a_n) \cdot \cdot \cdot \cdot (b_1 - a_1)$$

$$= \mathrm{Vol}(X)$$

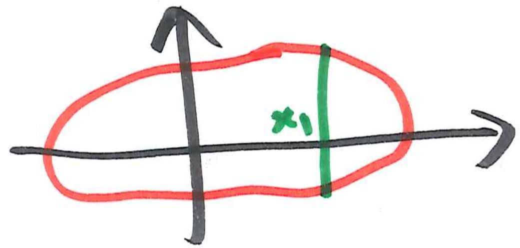
<u>Def</u> If $X \subset \mathbb{R}^n$ closed and bounded then

$$Vol(X) = \int_X dx_1 \dots dx_n$$

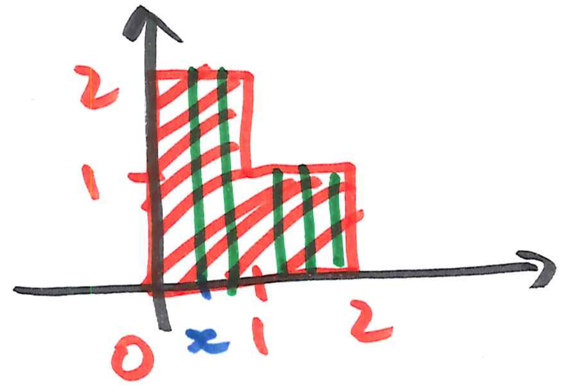
(<u>n-dim.</u> volume)

(<u>n = 2:</u> area)

<u>Ex. 2</u>: The function $g(x_1) = \int_{X_{x_1}} f(x_1, x_2)\, dx_2$ may not be continuous



$\underline{n=2}$, $f=1$



X is closed + bounded

$n = 2 = 1+1$

$X_1 = [0,2]$

$$X_x = \begin{cases} [0,2] & \text{if } 0 \le x \le 1 \\ [0,1] & \text{if } 1 < x \le 2 \end{cases}$$

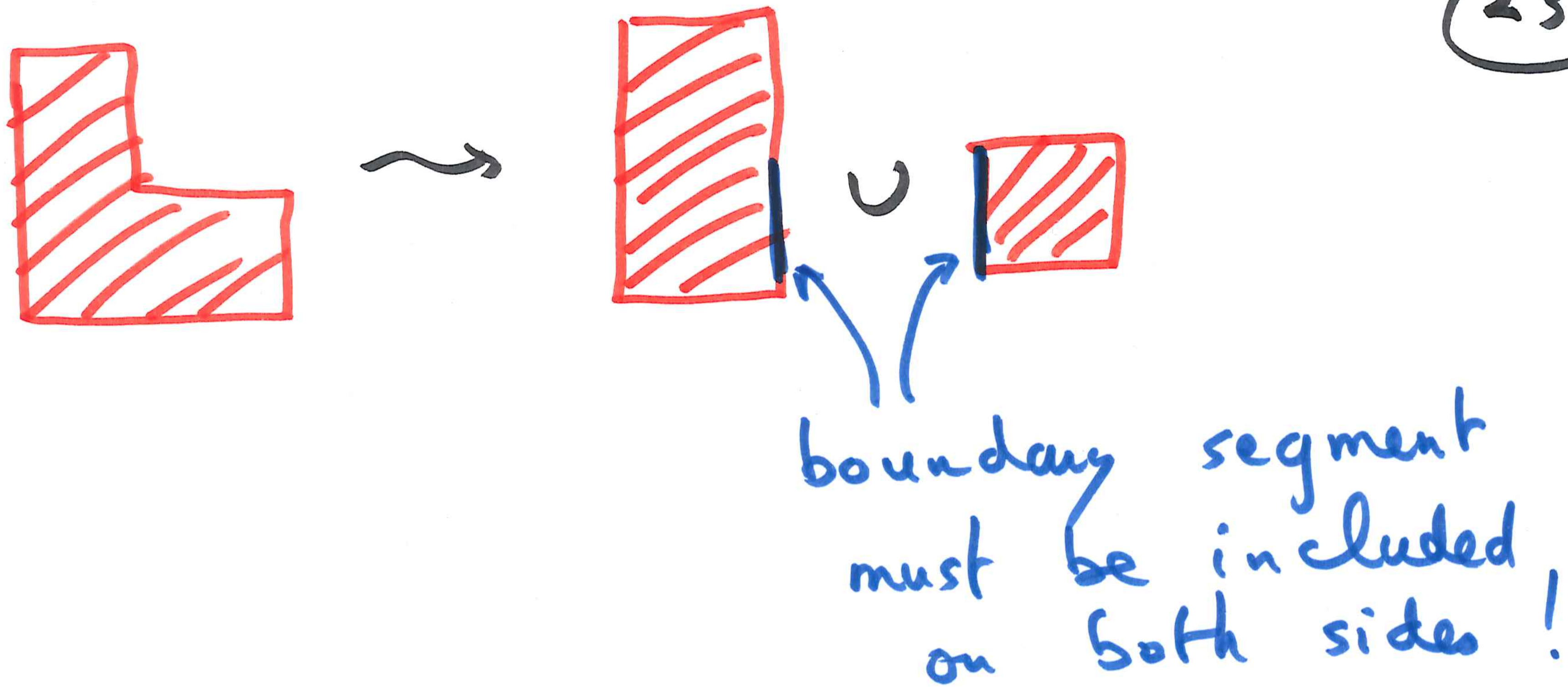
$$\Rightarrow \quad g(x) = \text{length of } X_x = \begin{cases} 2, & x \le 1 \\ 1, & 1 < x \le \end{cases}$$

is not continuous.

Idea to work around this problem: decompose the set X in smaller sets for which the function g <u>is</u> continuous.

(<u>Difficulty</u>: often the obvious decomposition $X = Y_1 \cup Y_2$, $(X_1 \cap X_2 = \emptyset)$ does not have all pieces Y_1, Y_2 closed.)

<u>Ex.</u>



Such intersection are "negligible" because smaller-dimensional

Property of the integral:

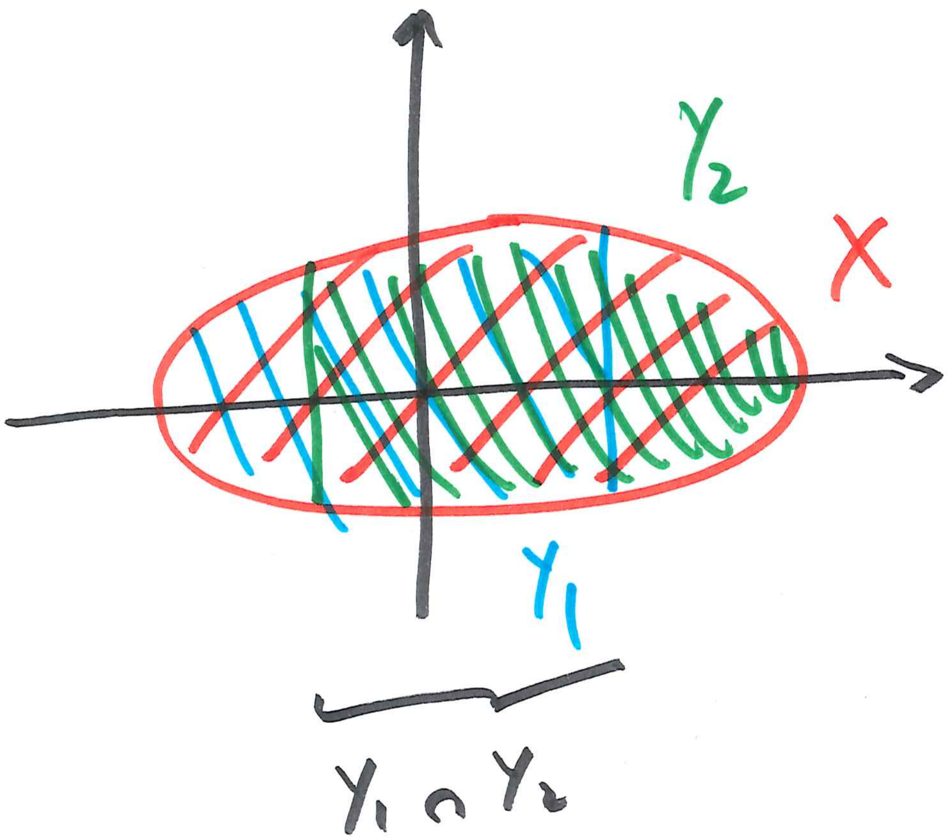
(6) [Domain additivity]

If $X = Y_1 \cup Y_2$ with Y_1, Y_2 bounded and closed, then for

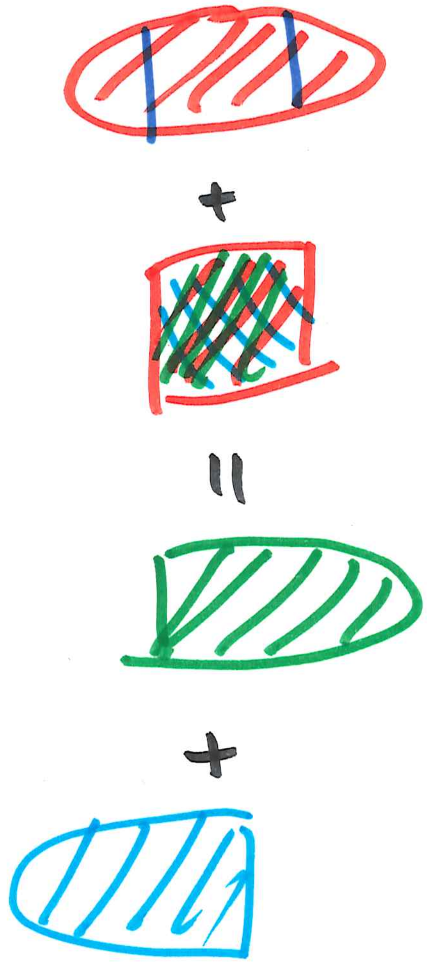
$$f : X \longrightarrow \mathbb{R}$$

we have

$$\int_{\underset{Y_1 \cup Y_2}{\overset{\shortparallel}{X}}} f + \int_{Y_1 \cap Y_2} f = \int_{Y_1} f + \int_{Y_2} f$$



$f = 1$



$$\boxed{\int_{Y_1 \cup Y_2} f + \int_{Y_1 \cap Y_2} f = \int_{Y_1} f + \int_{Y_2} f}$$

In particular, if $\int_{Y_1 \cap Y_2} f = 0$

then

$$\int_{Y_1 \cup Y_2} f = \int_{Y_1} f + \int_{Y_2} f$$

Fact: if $\mathrm{Vol}_n(Y_1 \cap Y_2) = 0$

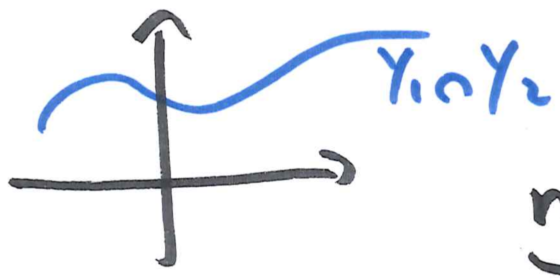
Vol_n — n-dim. vol

then $\int_{Y_1 \cap Y_2} f \, dx_1 \dots dx_n = 0$ for **any** f

Ex. (intuitively)

n = 2: $Y_1 \cap Y_2$ has area 0

e.g. a **curve**



n = 3: $Y_1 \cap Y_2$ is a surface

<u>Def.</u> (4.2.3) $\mathbb{R}^n$

(1) For $1 \leq m \leq n$, an <u>m-parameterized set</u> in $\mathbb{R}^n$ is a continuous

$$f : [a_1, b_1] \times \cdots \times [a_m, b_m] \longrightarrow \mathbb{R}^n$$

which is $\underline{\underline{\underline{C^1}}}$ on $]a_1, b_1[\times \cdots \times]a_m, b_m[$.

[<u>ex.</u> $m = 1$: a parameterized curve

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n \,]$$

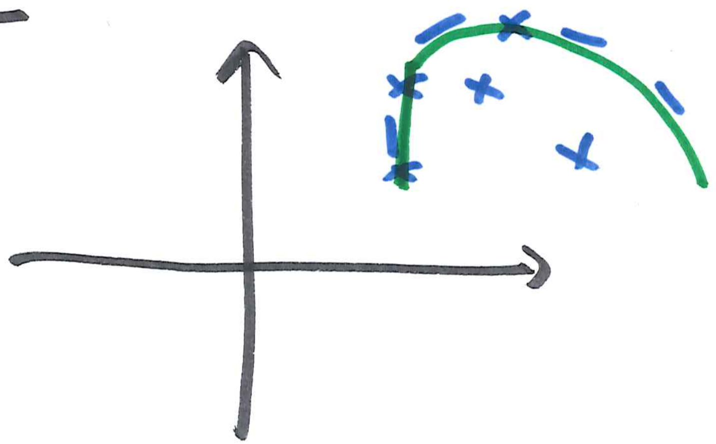
(2) A set $Y \subset \mathbb{R}^n$ is <u>negligible</u>

if there are finitely many f_i m_i - param. sets with $m_i < n$ s.t.

$$Y \subset \bigcup f_i(X_i)$$

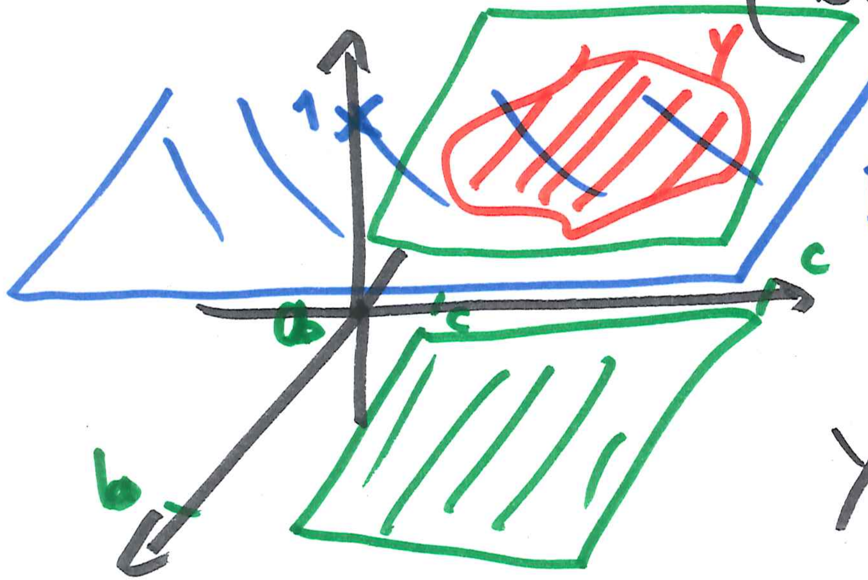
where $f_i : X_i \longrightarrow \mathbb{R}^n$

Ex. $\underline{n=2}$: $Y \subset$ union of finitely images of parameterized curves



n = 3: $Y \subset$ plane in $\mathbb{R}^3$

(bounded and closed)



Plane $z = 1$

$$Y \subset \left\{ (x, y, 1) \;\middle|\; \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix} \right\}$$

$=$ 2-parameterized

$$\left\{ \begin{array}{ccc} (x,y) & \longmapsto & (x,y,1) \\ [a,b] \times [c,d] & \longrightarrow & \mathbb{R}^3 \end{array} \right.$$

Property: if $Y \subset \mathbb{R}^n$ is closed, bounded, negligible

then $$\boxed{\int_Y f(x_1, \dots, x_n)\, dx_1 \dots dx_n = 0}$$

for all $f : Y \longrightarrow \mathbb{R}$ continuous.