Notation: higher partial derivatives

$f \in C^k(X, \mathbb{R}^m)$, $k \geq 2$ off

let $1 \leq m \leq k$; a partial derivative of order m is determined by:

how many times f is differentiated with respect to x_1, say $m_1 \geq 0$

————————————— x_2, say $m_2 \geq 0$

...

————————————— x_n, say $m_n \geq 0$

where $m_1 + \cdots + m_n = m$.

<u>Notation</u>: $\underline{m} = (m_1, \ldots, m_n)$

$$\frac{\partial^m f}{\partial x_1^{m_1} \cdot \cdots \cdot \partial x_n^{m_n}} = \partial_{(m_1, \ldots, m_n)} f$$

(x_i omitted if $m_i = 0$)

$$= \partial^m_{\underline{x}^{\underline{m}}} f$$

vector

$$(x_1, \ldots, x_n)^{(m_1, \ldots, m_n)} = (x_1^{m_1}, \ldots, x_n^{m_n})$$

Ex. $n = 3$ (x, y, z)

$$\partial_{x^2 y z^3}(f) = \frac{\partial^6 f}{\partial x^2 \partial y \partial z^3}$$

(twice x
once y
three times z)

Recall: $f \in C^2$

$$\mathrm{Hess}_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i, j \le n}$$

(symmetric matrix)

3.6 – Change of variable

Idea:

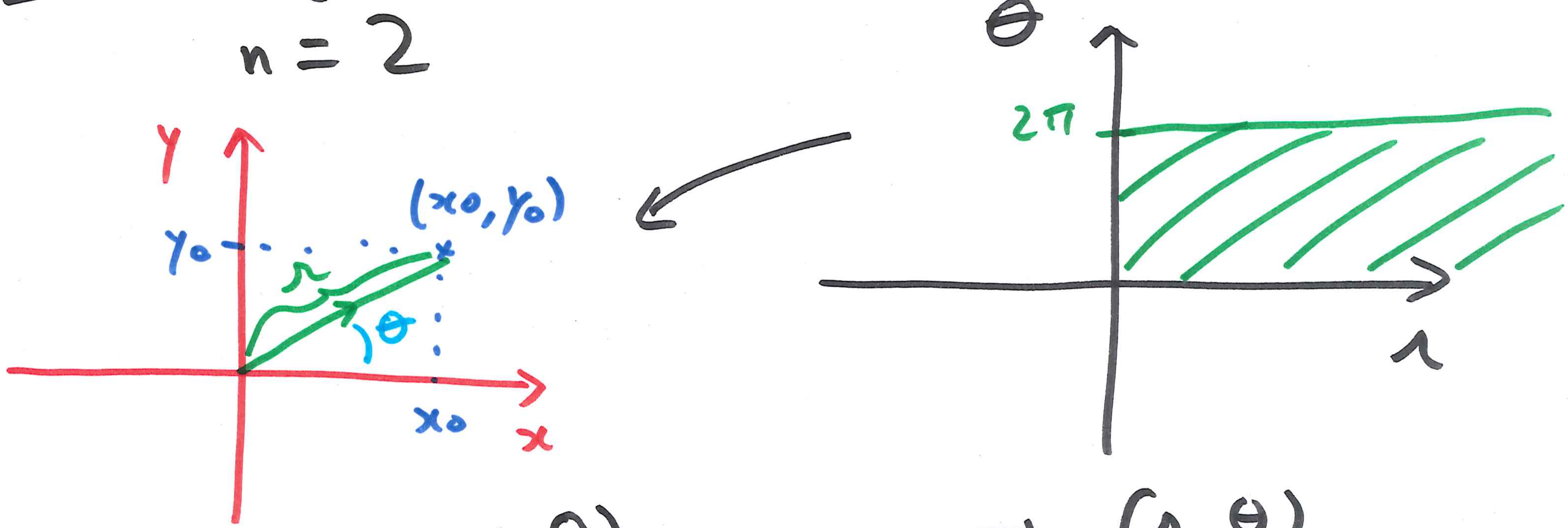
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, \dots, x_n)$$

Maybe f has a simpler expression in "different" coordinates (for instance if f is linear $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and diagonalizable).

We then want to use the "new" coordinates to compute with f.

<u>Ex.</u> (Polar coordinates)

$n = 2$



$$(r\cos\theta, r\sin\theta) \longleftarrow (r, \theta)$$

$$f(x,y) = \cos(x^2+y^2-1)$$

$$= \cos(r^2-1)$$

in polar coordinates

Formalization:

(g expresses each variable x_i in terms of "new" variables $(y_1,\dots,y_n)$)

$$\begin{array}{lll} U \subset \mathbb{R}^n \text{ open} & & (x_1,\dots,x_n) \\ g \uparrow & & \\ V \subset \mathbb{R}^n \text{ open} & & (y_1,\dots,y_n) \end{array}$$

(bijection)

141

Ex. $(x,y) \in \mathbb{R}^2$ (rectangular)

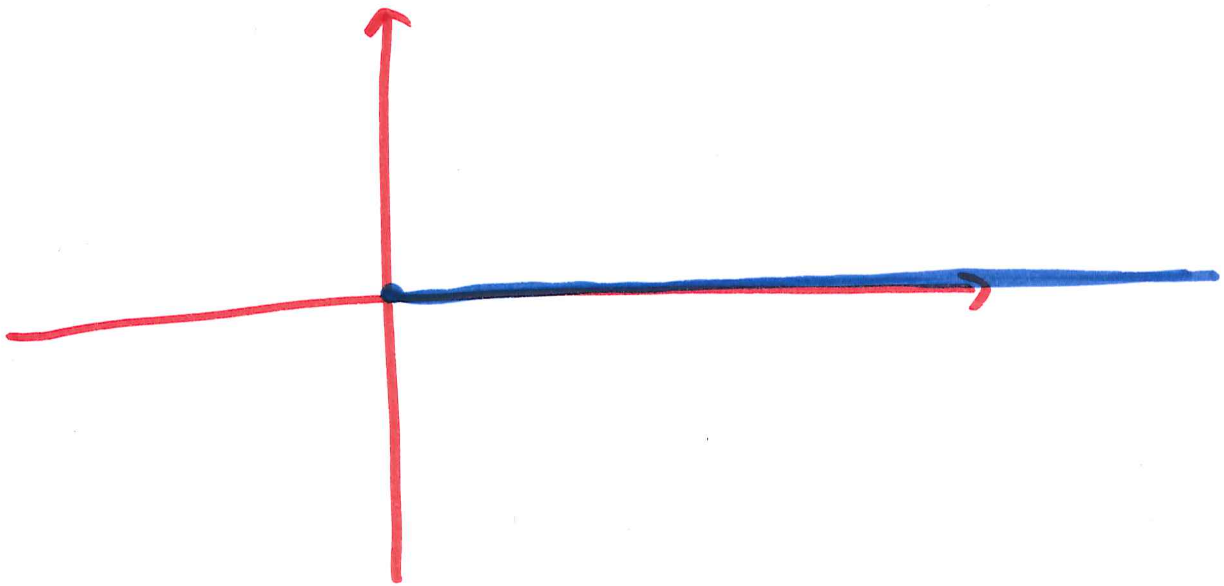
$(r\cos\theta, r\sin\theta) \uparrow g$

$(r,\theta) \in \,]0,+\infty[\times]0,2\pi[$ (polar)

Note: we have to adapt a bit the definition sets to have bijections:

$$]0,+\infty[\times]0,2\pi[\xrightarrow{g} U \subset \mathbb{R}^2$$

$$U = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (x,0) \text{ with } x \geq 0\}$$



(It turns out that the difference between U and $\mathbb{R}^2$ is not too important.)

$$V \xrightarrow{g} U \xrightarrow{f} \mathbb{R}$$

$f \circ g$ is "the function f in the new variables"

<u>ex.</u> Polar coord. ; $f(x,y) = \cos(x^2+y^2-1)$

$$\longrightarrow (f \circ g)(r,\theta) = \cos(r^2-1).$$

We can compute partial derivatives of $f \circ g$ using the chain rule.

<u>Remark:</u> g should be thought of as being "given" and well-known, including differentiability, formulas for partial derivatives, etc. The data that changes is <u>f</u>.

Let $h = f \circ g$

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[rr, bend right, "h"'] & U \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

By Chain Rule:

$$\frac{\partial h}{\partial y_1}(y) = \frac{\partial f}{\partial x_1}(g(y)) \cdot \frac{\partial g_1}{\partial y_1}(y) + \cdots\cdots + \frac{\partial f}{\partial x_n}(g(y)) \cdot \frac{\partial g_n}{\partial y_1}(y)$$

where $g(y_1, \ldots, y_n) = (g_1, \ldots, g_n)$,

for $y \in V$.

(This gives the partial derivatives in the new variables.)

Notation:

- Replace h by f ("because h is really f in different coordinates")

[ex. f is "distance to the origin" or other description that does not use a formula]

$\dfrac{\partial f}{\partial y_1}$ means "differentiate this abstract function with respect to y_1"

- Replace g_i by x_i

<u>Write</u>:

$$\boxed{\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_n}}$$

(This is how these formulas are written in books).

One can then also solve for $\frac{\partial f}{\partial x_i}$ in terms of $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$.

Example - (Polar coordinates)

Goal: explain how to compute

$$\Delta f = \partial_{x^2} f + \partial_{y^2} f$$

in terms of polar coordinates.

$$g(r, \theta) = (r\cos\theta, r\sin\theta)$$

$$J_g(r, \theta) = \begin{array}{c} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ \begin{array}{cc} \partial_r \qquad\qquad & \partial_\theta \end{array} \end{array}$$

(148)

$$\left[\ J_g = \begin{pmatrix} \frac{1}{r}x & -y \\ \frac{1}{r}y & x \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix} \right]$$

Given $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$J_f(x,y) = \begin{pmatrix} \partial_x f & \partial_y f \end{pmatrix}$$

$$h = f \circ g : \underset{]0,+\infty[\,\times\,]0,2\pi[}{\overset{\shortparallel}{\cancel{V}}} \longrightarrow \mathbb{R}$$

$$J_h = J_f \circ J_g = \begin{pmatrix} \partial_x f & \partial_y f \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$(\partial_r h, \partial_\theta h) = \left(\cos\theta\, \partial_x f + \sin\theta\, \partial_y f \;\middle|\; -r\sin\theta\, \partial_x f + r\cos\theta\, \partial_y f \right)$$

e.g

$$\begin{cases} \partial_r h = \cos\theta\, \partial_x f + \sin\theta\, \partial_y f = \partial_r f \\ \partial_\theta h = -r\sin\theta\, \partial_x f + r\cos\theta\, \partial_y f = \partial_\theta f \end{cases}$$

We can solve these in terms of $\partial_r f$ and $\partial_\theta f$:

$$\begin{cases} \partial_x f = \cos(\theta)\, \partial_r f - \frac{1}{r} \sin(\theta)\, \partial_\theta f \\ \partial_y f = \sin(\theta)\, \partial_r f + \frac{1}{r} \cos(\theta)\, \partial_\theta f \end{cases}$$ 150

<u>Concretely:</u> if one knows f in polar coordinates, then these formulas say what is $\partial_x f$ in terms of polar coordinates.

<u>Ex.</u> $f(x,y) = x^2 + y^2$ ("square of distance to $(0,0)$")

151

In polar coordinates:

$$f = r^2$$

First formula:

$$2x = \partial_x f = \cos(\theta) \cdot 2r - \frac{1}{r}\sin(\theta) \cdot 0$$
$$= 2r\cos(\theta) = 2x$$

Back to $\Delta f = \partial_{x^2} f + \partial_{y^2} f$

$$\partial_{x^2} f = \partial_x(\underline{\partial_x f}) \cancel{= \partial_x\left(\cos\theta\, \partial_r f - \frac{1}{r}\sin\theta\, \partial_\theta f\right)}$$
$$= \cos\theta\, \partial_r(\underline{\partial_x f}) - \frac{1}{r}\sin\theta\, \partial_\theta(\underline{\partial_x f})$$
$$= \cos(\theta)\, \partial_r\left(\underline{\cos(\theta)\partial_r f - \frac{1}{r}\sin(\theta)\partial_\theta f}\right)$$

$$- \frac{1}{r} \sin(\theta)\, \partial_\theta \Big(\cos(\theta)\, \partial_r f - \frac{1}{r} \sin(\theta)\, \partial_\theta f \Big)$$ 152

$$= \cos^2(\theta)\, \partial_{r^2}(f) - \cos\theta \sin\theta \Big(-\frac{1}{r^2} \partial_\theta f + \frac{1}{r} \partial^2_{r\theta} f \Big)$$

$$- \dots \dots$$

$= ($~~see~~ script$)$

Same for $\partial_{y^2} f = \dots \dots$

and

$$\boxed{\Delta f = \partial_{r^2} f + \frac{1}{r} \partial_r f + \frac{1}{r^2} \partial_{\theta^2} f}$$

Ex.

$$f = r^3 = (x^2+y^2)^{3/2}$$

$$\Delta f = 6r + 3r = 9r = 9\sqrt{x^2+y^2}$$

Summary: one shouldn't worry (too much) about doing this type of computations (Δf in polar coords.)

But one should be very comfortable applying these formulas.

3.7 - Taylor polynomials

Goal: (1) give better approximation to f close to a point than

$$f(x) \cong f(x_0) + \nabla f(x_0) \odot (x - x_0)$$

scalar product

$[f: \mathbb{R}^n \longrightarrow \mathbb{R}]$

(2) criterion for finding a local minimum or maximum to a function

$\left(\underline{n=1} \quad f'(x_0) = 0; \text{ compute sign of } f''(x_0) \right)$

Def. $f : U \longrightarrow \mathbb{R}$, U open

A <u>critical point</u> of f in U

is a point x_0 where $J_f(x_0) = 0$

$$\Updownarrow$$

$$\nabla f(x_0) = 0$$

$$\Updownarrow$$

$$\left(i = 1, \ldots, n \qquad \frac{\partial f}{\partial x_i}(x_0) = 0 \right)$$

[If x_0 is a local max/min, then it is a critical point.

$n=1$: Taylor polynomial at x_0 of degree $\leq k$ of $f \in C^m$, $m \geq k$:

$$T_k f(y\,;\,x_0) = \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}\, y^j$$

variable of the polynomial

For general n: more complicated because all possible derivatives

$$\frac{\partial^j f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) \quad \text{with } j = m_1 + \cdots + m_n \leq k$$

appear.