Multiple choice questions

(1) Let $f = (f_1, f_2)$ be a C^1 vector field on $\mathbf{R}^2$ and $\gamma\colon [0,1] \to \mathbf{R}^2$ a parameterized curve. The definition of the line integral of f along γ is

$$\int_\gamma f \cdot d\vec{s} = \int_0^1 f(\gamma(t))dt.$$

YES ▫ NO ▫

(2) Let $f = (f_1, f_2, f_3)$ be a C^1 vector field on $\mathbf{R}^3$ such that

$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}.$$

Is the vector field f conservative?

YES ▫ NO ▫

(3) Let S be the sphere centered at 0 of radius 1 in $\mathbf{R}^3$ and f continuous on $\mathbf{R}^3$. We have

$$\int_S f(x,y,z)dxdydz = \int_{-1}^1 \Big(\int_{-1}^1 \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dydz\Big)dx.$$

YES ▫ NO ▫

(4) Let D be the disc centered at $0 \in \mathbf{R}^2$ with radius $3/2$. For a continuous function f on $\mathbf{R}^2$, we have

$$\int_D f(x,y)dxdy = \int_0^{3/2} \Big(\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) r d\theta\Big) dr.$$

YES ▫ NO ▫

Quick computations

(1) For which values of $a \in \mathbf{R}$, is the vector field $f(x, y) = (xy^2 + ay^2, ay + x^2y)$ on $\mathbf{R}^2$ conservative?

(2) Let f be the vector field on $\mathbf{R}^2$ defined by

$$f(x, y) = \Big(\frac{2x}{(x^2 + y^4 + 1)^2}, \frac{4y^3}{(x^2 + y^4 + 1)^2}\Big).$$

Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

where $\gamma(t) = (\sin(\pi \cos(2\pi t)), (1 - t)e^t + t)$ for $0 \leqslant t \leqslant 1$.

(3) Compute the integral

$$\int_S (x^2 + y^2 + y^2)^2 dxdydy$$

where S is the half of the sphere in $\mathbf{R}^3$ centered at $(0, 0, 0)$ with radius 1 where $z \geqslant 0$.

(4) Let $f(x, y) = (\cos(x + y), x^2)$. Compute the line integrala

$$\int_\gamma f \cdot d\vec{s}$$

along the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$ oriented counter-clockwise.

Solutions to multiple choice questions

(1) Let $f = (f_1, f_2)$ be a C^1 vector field on $\mathbf{R}^2$ and $\gamma\colon [0,1] \to \mathbf{R}^2$ a parameterized curve. The definition of the line integral of f along γ is

$$\int_\gamma f \cdot d\vec{s} = \int_0^1 f(\gamma(t))dt.$$

NO – As the dot in the notation indicates, the correct definition is

$$\int_\gamma f \cdot d\vec{s} = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt = \int_0^1 \Big(f_1(\gamma(t))\gamma_1'(t) + f_1(\gamma(t))\gamma_2'(t) \Big) dt.$$

(In fact, the right-hand side of the proposed formula is a vector, and not a number, as the line integral is supposed to be).

(2) Let $f = (f_1, f_2, f_3)$ be a C^1 vector field on $\mathbf{R}^3$ such that

$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}.$$

Is the vector field f conservative?

YES – The three equations are the three coordinates (in some order) of the curl vector field, and since $\mathbf{R}^3$ is star-shaped (even convex), the vanishing of the curl on $\mathbf{R}^3$ is equivalent to being conservative.

(3) Let S be the sphere centered at 0 of radius 1 in $\mathbf{R}^3$ and f continuous on $\mathbf{R}^3$. We have

$$\int_S f(x,y,z)dxdydz = \int_{-1}^1 \Big(\int_{-1}^1 \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dydz \Big) dx.$$

NO – The correct formula for applying twice Fubini's Theorem is

$$\int_S f(x,y,z)dxdydz = \int_{-1}^1 \Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dydz \Big) dx.$$

(In fact, the right hand side of the proposed formula is not well-defined since $1 - x^2 - y^2$ could be negative if x and y are both allowed to vary independently between -1 and 1.)

(4) Let D be the disc centered at $0 \in \mathbf{R}^2$ with radius $3/2$. For a continuous function f on $\mathbf{R}^2$, we have

$$\int_D f(x,y)dxdy = \int_0^{3/2} \Big(\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) r d\theta \Big) dr.$$

YES – This is the change of variable formula for polar coordinates, where we take the angle to lie in the interval $[-\pi, \pi]$.

Solutions to quick computations

(1) For which values of $a \in \mathbf{R}$, is the vector field $f(x, y) = (xy^2 + ay^2, ay + x^2y)$ on $\mathbf{R}^2$ conservative?

Since $\mathbf{R}^2$ is star-shaped and f is C^1, the vector field is conservative if and only if

$$\partial_y(xy^2 + ay^2) = \partial_x(ay + x^2y),$$

which translates to the condition

$$2xy + 2ay = 2xy$$

which holds if and only if $a = 0$.

(2) Let f be the vector field on $\mathbf{R}^2$ defined by

$$f(x, y) = \Big(\frac{2x}{(x^2 + y^4 + 1)^2}, \frac{4y^3}{(x^2 + y^4 + 1)^2}\Big).$$

Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

where $\gamma(t) = (\sin(\pi \cos(2\pi t)), (1 - t)e^t + t)$ for $0 \leqslant t \leqslant 1$.

Observe that γ is a closed curve, since $\gamma(0) = (0, 1) = \gamma(1)$, and that f is C^1 and conservative because $f = \nabla g$, where $g(x, y) = -1/(x^2 + y^4 + 1)$. So the line integral is zero.

(3) Compute the integral

$$\int_S (x^2 + y^2 + y^2)^2 dxdydy$$

where S is the half of the sphere in $\mathbf{R}^3$ centered at $(0, 0, 0)$ with radius 1 where $z \geqslant 0$.

Using spherical coordinates, where the azimuthal angle is restricted to the interval from 0 to $\pi/2$, the integral is equal to

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} r^4 \times r^2 \sin(\varphi) dr d\theta d\varphi = \frac{1}{7} \times 2\pi \times \int_0^{\pi/2} \sin(\varphi) d\varphi = \frac{2\pi}{7}.$$

(4) Let $f(x, y) = (\cos(x + y), x^2)$. Compute the line integral

$$\int_\gamma f \cdot d\vec{s}$$

along the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$ oriented counter-clockwise.

We use Green's Theorem: since the path γ is the boundary, positively oriented, of the square $[0,1]^2$, the line integral is equal to

$$\int_{[0,1]^2} (\partial_x(x^2) - \partial_y(\cos(x+y)))dxdy = \int_0^1 \int_0^1 (2x + \sin(x+y))dxdy$$

$$= 1 + \int_0^1 x\Big(\int_0^1 \sin(x+y)dy\Big)dx = 1 + \int_0^1 (-\cos(x+1) + \cos(x))dx$$

$$= 1 + \Big(-\sin(2) + \sin(1) - \sin(1)\Big) = 1 - \sin(2).$$