D-MAVT
D-MATL

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Analysis III

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Serie 10

You need to know: Separation of variables, d'Alembert's solution of the 1-dimensional wave equation on $\mathbb{R}$, Fourier series solution of the 1-dimensional wave equation on an interval $[0, L]$.

1. Find all possible solutions of the following PDEs with variables separated, i.e. solutions of the form $u(x,t) = F(x)G(t)$:

a) $xu_x + u_t = 0$

b) $u_x + u_t + xu = 0$

c) $t^3 u_x + \cos(x)u - 2u_{xt} = 0$

2. Let $u(x,t)$ be the solution of the following problem (1-dimensional wave equation on the line).

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} e^{\frac{x^2}{x^2-1}}, & |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch a graph of $f(x)$, which is the solution at the initial time.

b) Sketch a graph of the solution at the time $t = 2$, $u(x,2)$.

c) Prove that, for each fixed $x \in \mathbb{R}$:

$$\lim_{t \to +\infty} u(x,t) = 0.$$

3. For $k \in \mathbb{R}$, find the Fourier series solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,1]$ with the following boundary and initial conditions:

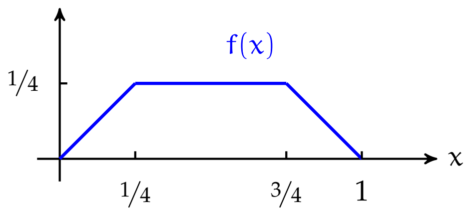
$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geqslant 0 \\ u(x,0) = kx(1-x^2), & 0 \leqslant x \leqslant 1 \\ u_t(x,0) = 0, & 0 \leqslant x \leqslant 1 \end{cases}$$

Use the method of separation of variables from scratch and <u>describe each step</u> of it.

4. Find (via Fourier series) the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,1]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant 1 \\ u_t(x,0) = 0, & 0 \leqslant x \leqslant 1 \end{cases}$$

where $f(x)$ is the following function



You can skip separation of variables and use the formula obtained in the previous exercise (and in Paragraph 4.3.3. in the Lecture notes).

5. Find (via Fourier series) the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,1]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(\pi,t), & t \geqslant 0 \\ u(x,0) = 0, & 0 \leqslant x \leqslant \pi \\ u_t(x,0) = g(x), & 0 \leqslant x \leqslant \pi \end{cases}$$

where

$$g(x) = \begin{cases} \frac{x}{100}, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi - x}{100}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}$$

You can use the formula from the Lecture notes.

<u>**Hand in by:**</u> Thursday 28 November 2019.