D-MAVT
D-MATL

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Analysis III

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Serie 11

You need to know: d'Alembert's solutions of the 1-dimensional wave equation: domain of dependence and region of influence. Fourier series solution of the 1-dimensional heat equation.

1. (Graphical exercise on the domain of dependence / region of influence)

Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} & x \in \mathbb{R} \\ u_t(x,0) = 0. & x \in \mathbb{R} \end{cases}$$

a) Draw the characteristic lines as in the picture at pag. 59 of the Lecture notes for the interval $[a,b] = [-1,1]$. You should have divided the upper-half plane $(x,t) \in \mathbb{R} \times [0,+\infty)$ into the six regions denoted in the picture by I,II,III,IV,V,VI.

b) In this case in each of these regions, and on each of the characteristics line, the solution is equal to a constant. Find all these constants and write them down in the regions and on the lines.

c) In this way you have found graphically the solution for each (x,t). What are the maximum and minimum values?

- $\max\limits_{(x,t)\in\mathbb{R}\times[0,+\infty)} u(x,t) = ?$
- $\min\limits_{(x,t)\in\mathbb{R}\times[0,+\infty)} u(x,t) = ?$

2. Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0. & |x| > 1 \end{cases} & x \in \mathbb{R} \end{cases}$$

a) Find the values $u(0, \frac{1}{2})$ and $u(\frac{3}{2}, \frac{1}{2})$.

b) Find, for each fixed $x \in \mathbb{R}$, the asymptotic limit

$$\lim_{t \to +\infty} u(x,t).$$

3. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\, u_{xx}, & x \in [0,1],\ t \geqslant 0 \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

Use the method of separation of variables from scratch, showing all the steps.

4. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001 \left(\frac{\mathrm{m}^2}{\mathrm{sec}}\right) = 10^{-4} \left(\frac{\mathrm{m}^2}{\mathrm{sec}}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100 \sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature $\leqslant$ 30°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4} u_{xx}, & x \in [0,1],\ t \geqslant 0 \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = 100 \sin(\pi x), & x \in [0,1]. \end{cases}$$

and find the smallest t^* for which

$$\max_{x \in [0,1]} u(x, t^*) \leqslant 30.$$

You can use the formula from the Lecture notes (pag. 61).

Hand in by: Thursday 5 December 2019.

[1] we are approximating the standard value which would be $c^2 \approx 0.000097 \mathrm{m}^2/\mathrm{sec}$ to make computations easier.