D-MAVT
D-MATL

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Analysis III

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Serie 12

You need to know: normal form of a hyperbolic PDE, Laplace equation on a rectangle, solution of the heat equation on an infinite bar (via Fourier integral and Fourier transform).

1. Consider the following hyperbolic PDE, transform it into normal form, and solve it.

$$u_{xx} + 2u_{xy} - 3u_{yy} = e^{x+2y}$$

2. Solve the following Laplace equation (steady heat equation) on the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2\},$$

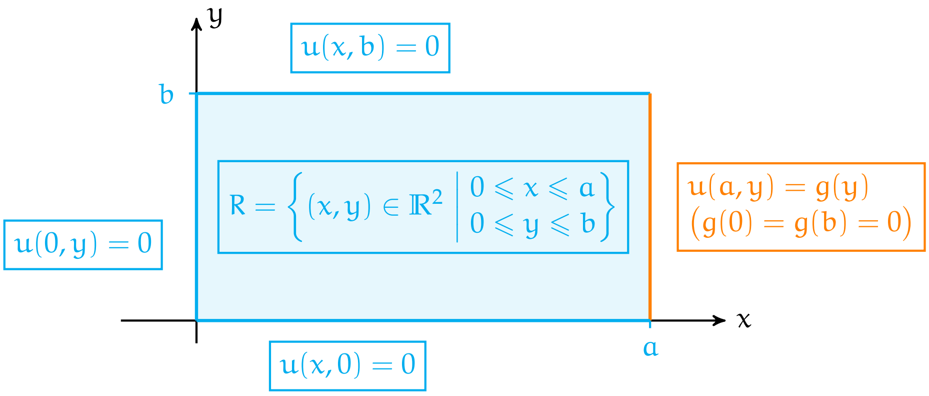
$$\begin{cases} \Delta u = 0, & (x, y) \in R \\ u(0, y) = u(1, y) = 0, & 0 \leqslant y \leqslant 2 \\ u(x, 0) = 0, & 0 \leqslant x \leqslant 1 \\ u(x, 2) = f(x), & 0 \leqslant x \leqslant 1 \end{cases}$$

where

$$f(x) = x(1 - x).$$

You can use the formula from the Lecture notes.

3. Adapt the method used to solve the previous Laplace equation in the case in which the only nontrivial initial boundary condition is on the vertical segment on the right of the rectangle



$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = u(x,b) = 0, & 0 \leqslant x \leqslant a \\ u(0,y) = 0, & 0 \leqslant y \leqslant b \\ u(a,y) = g(y), & 0 \leqslant y \leqslant b \end{cases}$$

where $g(y)$ is any function with prescribed boundary conditions

$$g(0) = g(b) = 0.$$

4. Find the solution of the heat equation on an infinite bar

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = f(x) = \begin{cases} \sinh(x), & |x| \leqslant 1 \\ 0, & \text{otherwise} \end{cases} & x \in \mathbb{R} \end{cases}$$

in Fourier integral form - formula (4.24) of the Lecture notes.

5. **Exercise from previous Exam**
Solve the following heat equation on an infinite bar:

$$\begin{cases} u_t = \frac{1}{2} u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = xe^{-\frac{1}{2}x^2}, & x \in \mathbb{R} \end{cases}$$

via the Fourier transform with respect to x.
Hint: Use that, for $a > 0$,

$$\mathcal{F}\left(xe^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}} e^{-\frac{\omega^2}{4a}}.$$

Hand in by: Thursday 12 December 2019.