D-MAVT
D-MATL

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Analysis III

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Serie 13

You need to know: Derivatives and Laplace operator in polar coordinates, Laplace equation on a disk with prescribed boundary conditions (Dirichlet problem).

Notation: We use the following notation for the centered disk of radius $R > 0$ and its boundary:

$$D_R := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant R^2\right\}$$
$$\partial D_R := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = R^2\right\}$$

1. Find the solution of the following Laplace equation on the disk of radius 2:

$$\begin{cases} \nabla^2 u = 0, & (x, y) \in D_2 \\ u(x, y) = x^3. & (x, y) \in \partial D_2 \end{cases}$$

Do as follows:

a) Write the boundary condition in polar coordinates.

b) Solve the problem in polar coordinates, using the methods/formulas from the Lecture notes.

[*Hint:* It might be useful at some point the trigonometric formula

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta) \,]$$

c) Express the solution in the standard cartesian coordinates:

$$u(x, y) = ?$$

2. **a)** Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \nabla^2 u = 0, & 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant 2\pi \\ u(R,\theta) = \sin^2(\theta). & 0 \leqslant \theta \leqslant 2\pi \end{cases}$$

[*Hint:* Remember the trigonometric formula

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)]$$

b) Find the maximum of $u(r,\theta)$. In which point(s) is it reached?

c) Express the solution in the standard cartesian coordinates.

3. Let u be the unique harmonic function ($\nabla^2 u = 0$) on the unit disk D_1 which on the boundary is

$$u(x,y) = xy + 3, \quad (x,y) \in \partial D_1.$$

Without computing any integral or using any formula from the script, answer the following questions:

a) Find $u(x,y)$.

b) Find the value in the center of the disk: $u(0,0)$= ?

c) Find the maximum of $u(x,y)$ and in which point(s) it is reached. Can you notice a similarity with the maximum points in the previous exercise (**2.b**)? What do they have in common?

4. Prove, without computing explicitely the integrals, that for each $0 \leqslant r < 1$ and for each $0 \leqslant \theta \leqslant 2\pi$:

a)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, d\phi = 1.$$

b)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\phi)\sin(\phi) - \sin^3(\phi)\cos(\phi)\big)}{1-2r\cos(\theta-\phi)+r^2}\, d\phi = \frac{r^4}{4}\sin(4\theta).$$