D-MAVT
D-MATL

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Analysis III

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Serie 14

<u>You need to know:</u> Harmonic functions. Mean value property and maximum(/minimum) principle.

1. Let u the unique harmonic function on the disk of radius R which on the boundary is

$$u(x, y) = x^2 y^2, \qquad (x, y) \in \partial D_R.$$

Answer, *without finding explicitely the function on the whole disk*, the following questions.

a) Find the value in the center of the disk:

$$u(0, 0) = \ ?$$

b) Find the maximum of u on the disk:

$$\max_{(x,y)\in D_R} u(x, y) = \ ?$$

c) Same question for the minimum.

2. Answer the following questions.

a) Let $a, b \in \mathbb{N}(= \{0, 1, 2, \ldots\})$ and u the solution of the following Laplace equation:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u = x^a y^b, & \partial D_R \end{cases}$$

For which values of a, b is it true that $u(0, 0) = 0$?

<u>Hint:</u> You should find that the answer depends on their parity.

b) Let u be the solution of the following Laplace equation:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(R,\theta) = 3R\, e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}. & \partial D_R \end{cases}$$

Find the only radius R for which

$$\max_{(x,y)\in D_R} u(x,y) = \pi$$

c) Let u be the solution of:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(R,\theta) = \sin^9(\theta). & \partial D_R \end{cases}$$

Is it true or false that $u + 1 \geqslant 0$ everywhere on the disk?.

3. Poisson's equation on a disk

Solve the following Poisson's equation on a disk of radius R:

$$\begin{cases} \nabla^2 u = x^2 + y^2, & D_R \\ u = 0, & \partial D_R \end{cases}$$

Proceed as follows.

a) Find a function g such that $\nabla^2 g = x^2 + y^2$.

b) Observe that the function $v := u - g$ solves the Laplace equation on the disk with some nonzero boundary condition. Write down this Dirichlet problem for v.

c) Solve this problem for v and then u will be $u = v + g$.