D-MAVT
D-MATL

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Analysis III

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Serie 15 - Ferienserie

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1. Using the Laplace transform, find the solution $y = y(t)$ of the following initial value problem:

$$\begin{cases} y''' + y'' = \delta(t-6), \quad t \geqslant 0 \\ y(0) = y'(0) = 0, \\ y''(0) = 1. \end{cases}$$

2. Let

$$f(x) = \begin{cases} x, & 0 \leqslant |x| \leqslant 1 \\ -\operatorname{sign}(x), & 1 < |x| < 2 \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch the graph of this function, in the interval $x \in [-3, 3]$.

b) Let

$$F(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{\big(\sin(\omega) - 2\omega\cos(\omega) + \omega\cos(2\omega)\big)\sin(\omega x)}{\omega^2}\, d\omega.$$

Prove that

$$\forall x \in \mathbb{R},\ x \neq \pm 1, \pm 2, \qquad F(x) = f(x).$$

c) What are the values of the integral in the remaining 4 points?

(i) $F(-1) = ?$
(ii) $F(1) = ?$
(iii) $F(-2) = ?$
(iv) $F(2) = ?$

3. Consider a string of length $L = \pi$, whose waves propagate with speed $c = 1$.
Let $u(x,t)$ be the height of the string at time $t \geqslant 0$ and point $x \in [0,\pi]$.
The string is initially motionless (zero initial speed), has initial shape given by

$$u(x,0) = x(x-\pi)$$

and fixed endpoints at height zero.

a) Formulate the problem whose solution is $u(x,t)$.

b) Find the solution. You are allowed to use the formula from the script.

4. Let $c > 0$. Consider the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = e^{-x^2}\sin^2(x) + x, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2}. & x \in \mathbb{R} \end{cases}$$

a) Find the solution $u(x,t)$. You may use D'Alembert formula.
[Simplify the expression as much as possible: no unsolved integrals].

b) For a fixed $a \in \mathbb{R}$, determine the asymptotic limit

$$\lim_{t \to +\infty} u(a,t).$$

5. **a)** We recall that for every $a > 0$ the Fourier transform of the Gaussian e^{-ax^2} is given by

$$\mathcal{F}\left[e^{-ax^2}\right](\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}. \tag{1}$$

Let $a > 0$ be fixed. From (1) deduce the integrals

(i) $\int_{\mathbb{R}} e^{-ax^2}\, dx$

(ii) $\int_{\mathbb{R}} x\, e^{-ax^2}\, dx$

(iii) $\int_{\mathbb{R}} x^2\, e^{-ax^2}\, dx$.

[You can use the formula

$$\mathcal{F}\left[x^k f(x)\right](\omega) = i^k \frac{d^k}{d\omega^k} \mathcal{F}\left[f(x)\right](\omega) \Big] \tag{2}$$

b) Recall that the solution $u = u(x,t)$ of the following problem

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases} \tag{3}$$

is given by

$$u(x,t) = \int_{-\infty}^{+\infty} f(y)K(x-y,t)\ dy, \tag{4}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{5}$$

is the so-called *Heat Kernel*.
Compute explicitely the solution of the problem

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = x^2. & x \in \mathbb{R} \end{cases} \tag{6}$$

6. Consider the following problem:

$$\begin{cases} u_t = c^2 u_{xx}, & x \in [0,\pi],\ t \geqslant 0 \\ u(0,t) = 2, & t \geqslant 0 \\ u(\pi,t) = 3, & t \geqslant 0 \\ u(x,0) = f(x), & x \in [0,\pi] \end{cases} \tag{7}$$

where

$$f(x) = \sin(x) - 3\sin(3x) + \frac{x}{\pi} + 2.$$

The boundary conditions are not homogeneous, therefore one cannot directly apply the formulas known. You should argue as follows:

a) Construct a function $w(x)$ with $w(0) = 2$, $w(\pi) = 3$ and $w'' = 0$.

b) Let u be a solution of the above problem (7). State the corresponding problem solved by the function $v(x,t) := u(x,t) - w(x)$.

c) Solve the problem for v using the method of separation of variables from scratch. Show all the steps of the method of separation of variables.

d) Find the solution u of the original problem (7) .

7. Let $D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ be the centered disk of radius 1 and $u = u(x,y)$ the solution of the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u(x,y) = x^5. & \text{on } \partial D_1 \end{cases}$$

a) Find the value in the center of the disk, $u(0,0)$.

b) Find the maximum of u on the whole disk,

$$\max_{(x,y)\in D_1} u(x,y).$$

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