D-MAVT
D-MATL

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Analysis III

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Serie 2

<u>You need to know:</u> Definition of Laplace transform. First properties: linearity, s-shifting. Inverse Laplace transform.

1. Find the Laplace transform $F(s) := \mathcal{L}(f)(s)$ of the following functions:

a) $f(t) = t^2 + 4t + 1$

b) $f(t) = \dfrac{1}{\sqrt{t}}$, using that

$$\Gamma\left(\frac{1}{2}\right)\left(= \int_0^{+\infty} t^{-1/2} e^{-t}\, dt\right) = \sqrt{\pi}$$

c) $f(t) = \sin(\omega t), \quad \omega \in \mathbb{R}$

d) $f(t) = \cos(\omega t), \quad \omega \in \mathbb{R}$

e) $f(t) = \sin(\alpha t)\cos(\beta t), \quad \alpha, \beta \in \mathbb{R}$

<u>Hint:</u> Exercise **e)** is much simplified using the alternative expression for that function found in Exercise **3.b)** of Serie 1.

Please turn!

2. Given a function $f(t)$ denote its Laplace transform by $F(s) := \mathcal{L}(f(t))(s)$.

a) If not already done in the lecture, prove that for functions $f(t)$ for which both their Laplace transform and the Laplace transform of $tf(t)$ is well-defined:

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}F(s). \tag{1}$$

b) Prove that for functions $f(t)$ for which both their Laplace transform and the Laplace transforms of $t^i f(t)$ (for all indices $i = 0, 1, \dots, n$) are well-defined:

$$\mathcal{L}(t^n f(t))(s) = (-1)^n \frac{d^n}{ds^n}F(s). \tag{2}$$

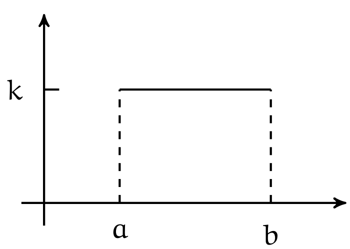
<u>*Hint:*</u> Iterate the formula from **a)**.

c) Choose $f(t) = 1$ and use **b)** to obtain another proof of

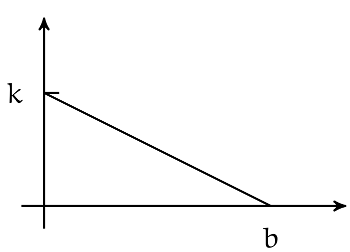
$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

3. Find the Laplace transform of the following functions:

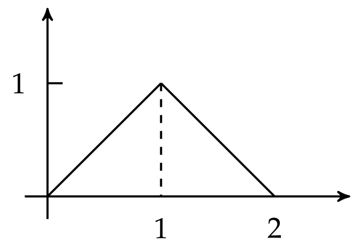
a)



b)



c)



4. Find the <u>inverse</u> Laplace transform $f = \mathscr{L}^{-1}(F)$ of the following functions:

a) $F(s) = \dfrac{1}{s^4}$

b) $F(s) = \dfrac{1}{(s-8)^{10}}$

c) $F(s) = \dfrac{s+3}{s^2-9}$

d) $F(s) = \dfrac{24}{(s-5)(s+3)}$

e) $F(s) = \dfrac{1}{s^2+4}$

f) $F(s) = \dfrac{1}{s^2+4s+20}$

g) $F(s) = \dfrac{s+1}{(s+2)(s^2+s+1)}$

h) $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$

<u>*Hint:*</u> For **f)** it may be useful to recognize that $s^2+4s+20$ can be written in the form $(s+a)^2+\omega^2$ for an opportune choice of a, ω. Then use the s-shifting property. The same technique is needed for **g),h)** but first it is necessary to use partial fraction decomposition to simplify the expression.

Please turn!

5. (Bonus exercise)

a) (For those who have never seen this)
Exercise **1.b)** assumes that $\Gamma(1/2) = \sqrt{\pi}$; this exercise proves it.
Let us call $I := \Gamma(1/2)$ this value:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t}\, dt.$$

(i) Use an opportune change of variables to prove that:

$$I = 2\int_0^{+\infty} e^{-x^2} dx.$$

(ii) Justify why

$$2\int_0^{+\infty} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx.$$

(iii) Compute this integral in a smart way by computing its square. Fill the dots (with some polar coordinates) to get

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy =$$
$$= \cdots = \pi$$

(iv) From (iii) we can deduce that the desired value is one of the two square roots of π: $I = \pm\sqrt{\pi}$.
Why can we exclude the negative value?

Theory reminder for Ex 5.b): The Laplace transform of a finite linear combination of functions is the linear combination of their Laplace transforms:

$$\mathcal{L}\left(a_1 f_1 + \cdots + a_m f_m\right) = a_1 \mathcal{L}(f_1) + \cdots + a_m \mathcal{L}(f_m), \quad a_1, \ldots, a_m \in \mathbb{R}.$$

The same thing is true for infinite linear combination of functions under opportune conditions of convergence (which are all satisfied in the following cases). For example we can compute the Laplace transform of the exponential from its power series expansion:

$$\mathcal{L}(e^{at})(s) = \mathcal{L}\left(\sum_{k=0}^{+\infty} \frac{(at)^k}{k!}\right)(s) = \sum_{k=0}^{+\infty} \frac{a^k}{k!} \mathcal{L}(t^k)(s) =$$

$$= \sum_{k=0}^{+\infty} \frac{a^k}{\cancel{k!}} \cdot \frac{\cancel{k!}}{s^{k+1}} = \frac{1}{s} \sum_{k=0}^{+\infty} \left(\frac{a}{s}\right)^k = \frac{1}{s} \cdot \frac{1}{1 - \frac{a}{s}} = \frac{1}{s - a}.$$

b) (i) Compute again $\mathcal{L}(\sin(\omega t))(s)$ using this method. Remember that:

$$\sin(\omega t) = \sum_{k=0}^{+\infty} (-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}$$

(ii) Prove that:

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \arctan\left(\frac{1}{s}\right).$$

Remember that the power series expansion of the arctangent is:

$$\arctan(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)}.$$

Hand in by: Thursday 3 October 2019.