D-MAVT
D-MATL

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Analysis III

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Serie 3

<u>You need to know:</u> Laplace transform of derivatives, Heaviside function, t-shifting theorem and applications to ODEs.

1. Find the Laplace transform of the following functions.

a) $f(t) = -3t^4 e^{-t/2}$

b) $f(t) = e^{-3t}\cos(\pi t)$

c) $f(t) = \cosh^2(t/2)$

d) $f(t) = t\cosh(3t)$

e) $f(t) = t^2 \sin(5t)$

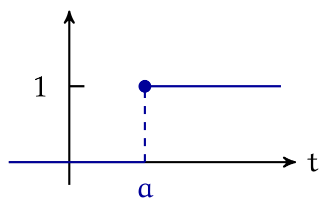
f) $f(t) = u(t-1)(t-1)^4$

g) $f(t) = u(t-\pi)\sin(t)$

h) $f(t) = -u(t-3)t^2 + u(t-5)\cos(t)$

<u>*Recall:*</u> For a fixed $a \in \mathbb{R}$, the function

$$u(t-a) = \begin{cases} 1, & t \geqslant a \\ 0, & t < a \end{cases}$$



is the Heaviside, or *unit step*, function (shifted by a).

2. First application to ODEs

Solve the following initial value problems with constant coefficients.

a) $$\begin{cases} y'' - y' - 6y = 0 \\ y(0) = 6 \\ y'(0) = 13 \end{cases}$$

b) $$\begin{cases} 2y'' + 3y' - 2y = te^{-2t} \\ y(0) = 0 \\ y'(0) = -2 \end{cases}$$

c) $$\begin{cases} y'' = 4 + u(t-2) \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$$

Recall: Under opportune hypothesis of regularity of the function y, the Laplace transform of its derivatives is:

$$\mathcal{L}\left(y^{(n)}\right)(s) = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0).$$

3. ODEs with nonconstant coefficients

Solve the following initial value problem with nonconstant coefficients.

$$\begin{cases} ty'' - ty' + y = 2 \\ y(0) = 2 \\ y'(0) = -4 \end{cases}$$

Hint: In this case the coefficients are nonconstant but very easy to transform when multiplied by a function. That is:

$$\mathcal{L}\left(ty'\right) = -\frac{d}{ds}\mathcal{L}\left(y'\right) = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s).$$
$$\mathcal{L}\left(ty''\right) = -\frac{d}{ds}\mathcal{L}\left(y''\right) = -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) = -s^2Y'(s) - 2sY(s) + y(0).$$

Euler-Bernoulli static beam equation

The Euler-Bernoulli beam equation describes - in a very simplified and elementary way - the deflection of a beam when subjected to some kind of load. It can be written as the following ODE of degree 4:

$$\frac{d^2}{dt^2}\left(EI\frac{d^2w}{dt^2}\right) = q.$$

$w(t)$ describes the deflection of the beam at the point $t \geqslant 0$ when a load is applied in the vertical direction. $q(t)$ is the distributed load, that is a force per unit length.[1] Finally, E and I depend on the material and the properties of the beam and their product $EI(t)$ is a measure of the resistance of the beam to deflect in the point t. Usually, the beam is assumed to be uniform, so that EI is a constant and the equation is simply

$$EI\frac{d^4w}{dt^4} = q.$$

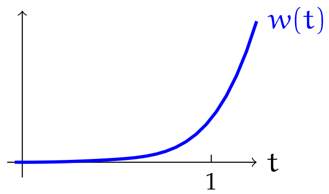
When possible to do it explicitly, this can be solved by integrating 4 times the function q. Another option we are interested in is to use the Laplace transform.

4. Prove that the solution of the following Euler-Bernoulli static beam equation

$$\begin{cases} \dfrac{d^4w}{dt^4} = u(t-1) \\ w(0) = w'(0) = 0 \\ w''(0) = 1 \\ w'''(0) = 0 \end{cases}$$

associated to a beam with a constant unit-strenght force applied from the point $t = 1$ on (corresponding to the Heaviside function $u(t-1)$), is given by:

$$w(t) = \tfrac{t^2}{2} + u(t-1)\tfrac{(t-1)^4}{4!}.$$



Observation: The shape of this function describes the final shape of the deflected beam under the force applied to it. We can thus see that the beam is deflected also in the points with coordinates $t \in [0,1]$, even if the force is applied only from $t = 1$, as a consequence of the fact that it has nonzero bending moment $w''(0) \neq 0$.

[1]the 1-dimensional analogue of the pressure = force per unit area.

5. (Bonus exercise)

a) Using the Laplace transform of the derivative, and the Laplace transform of $1/\sqrt{t}$ found in Exercise **1.b)** of Serie 2, prove that

$$\mathcal{L}\left(\sqrt{t}\right) = \frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

b) More generally, prove that for every integer $n \geqslant 1$:

$$\mathcal{L}\left(t^n\sqrt{t}\right) = \frac{(2n+2)!\sqrt{\pi}}{(4s)^{n+1}(n+1)!\sqrt{s}}.$$

Hand in by: Thursday 10 October 2019.