D-MAVT
D-MATL

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ETH Zürich
Autumn 2019

# Analysis III

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# Serie 4

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**You need to know:** Laplace transform of Dirac delta function, convolution product with applications to ODEs and integral equations.

**Theory reminder on convolution products:**
The convolution of two functions $f, g : [0, +\infty) \to \mathbb{R}$ is defined as the new function $f * g : [0, +\infty) \to \mathbb{R}$:

$$(f * g)(t) := \int_0^t f(\tau) g(t - \tau) \, d\tau = \int_0^t f(t - \tau) g(\tau) \, d\tau.$$

The useful property of the convolution product with respect to the Laplace transform is:

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g). \tag{1}$$

Therefore the inverse Laplace transform of a product is the convolution of the Laplace transforms:

$$\mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G). \tag{2}$$

**1.** Find the inverse Laplace transform of the following functions.

**a)** $F(s) = \dfrac{e^{-2s}}{s^2 + 4}$

**b)** $F(s) = \dfrac{e^{-s}}{(s + 1)^3}$

**c)** $F(s) = \dfrac{1}{s(s^2 + 1)}$

**d)** $F(s) = \dfrac{1}{(s^2 + 1)^2}$

**2.** Compute the following convolutions.

**a)** $e^{at} * e^{bt}, \qquad a, b \in \mathbb{R}$

**b)** $\sin(t) * \cos(t)$

**c)** $t^m * t^n, \qquad m, n \in \mathbb{N}$

*Hint:* Exercise **a)** requires a different discussion for the cases $a \neq b$ and $a = b$.

**3.** Solve the following initial value problem:

$$\begin{cases} y'' + 5y' + 6y = \delta(t - \frac{\pi}{2}) + u(t - \pi)\cos(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

**4. (Bonus exercise)**

**a)** Use the Laplace transform to find the solution of the following initial value problem:

$$\begin{cases} y' = g(t) \\ y(0) = c \end{cases}$$

**b)** Use the Laplace transform to find the solution of the following initial value problem:

$$\begin{cases} y'' + y = g(t) \\ y(0) = c \\ y'(0) = d \end{cases}$$

**Hand in by:** Thursday 17 October 2019.