D-MAVT
D-MATL

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ETH Zürich
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Analysis III

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Serie 5

You need to know: Fundamentals of Fourier analysis, periodic functions, trigonometric polynomials and trigonometric series. Fourier coefficients' formula.

Theory reminder on periodic functions:
A function $f : \mathbb{R} \to \mathbb{R}$ is periodic of period $P > 0$ if $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. A fundamental period is (if it exists) the smallest positive number P for which f is periodic of period P. For example for positive m, the functions $\sin(mx), \cos(mx)$ are periodic with fundamental period $\frac{2\pi}{m}$.

1. Determine if the following functions are periodic or not.
If they are, determine their fundamental period.
If they are not, give a proof/motivation why.

a) $f(x) = x$

b) $f(x) = \cos(4x) + \sin(3x)$

c) $f(x) = \sinh(x)$

d) $f(x) = \cos^4(2x)$

e) $f(x) = \sin(x^2)$

2. Determine whether the following functions are even, odd, or neither. Justify your answer.

a) $f(x) = x^2 + 2$

b) $f(x) = x + 1$

c) $f(x) = \sinh(x^3 + x)$

d) $f(x) = \sin(\pi x) + \sin(x^2)$

e) $f(x) = \mathfrak{Re}(e^{i\sin(x)})$

3. This exercise relates the periodicity of a function with its derivative(s) and its properties of boundedness.
Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

a) Prove that if f is periodic and continuous, then it is bounded.

b) Prove that if f is differentiable, and periodic of period P, then also f' is periodic with the same period.

c) From **a)** and **b)** deduce that if f is periodic and smooth (smooth, then it is bounded and all its derivative are bounded as well.

d) Use **c)** to give a very simple proof that $\sin(x^2)$ is not periodic.

4. This exercise relates the parity of a function with its derivatives.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions.

a) Prove that if f is even, then its derivative f' is odd.

b) Prove that if f is odd, then its derivative f' is even.

c) Suppose f, g are even. What can you say about the parity of $f + g$ and fg? Are they even, odd, or neither?

d) Same question with f, g odd.

5. **a)** Sketch the graph of the 2L-periodic extension of

$$f(x) = x, \quad x \in [-L, L)$$

in the interval $[-2L, 2L]$. In which points this extension is not continuous?

b) Sketch the graph of the 2-periodic, extension of

$$f(x) = \frac{1}{1 + x^2}, \quad x \in [-1, 1]$$

in the interval $[-3, 3]$. Is it continuous everywhere?

Hand in by: Thursday 24 October 2019.