D-MAVT
D-MATL

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Analysis III
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Solutions Serie 11

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1. (Graphical exercise on the domain of dependence / region of influence)

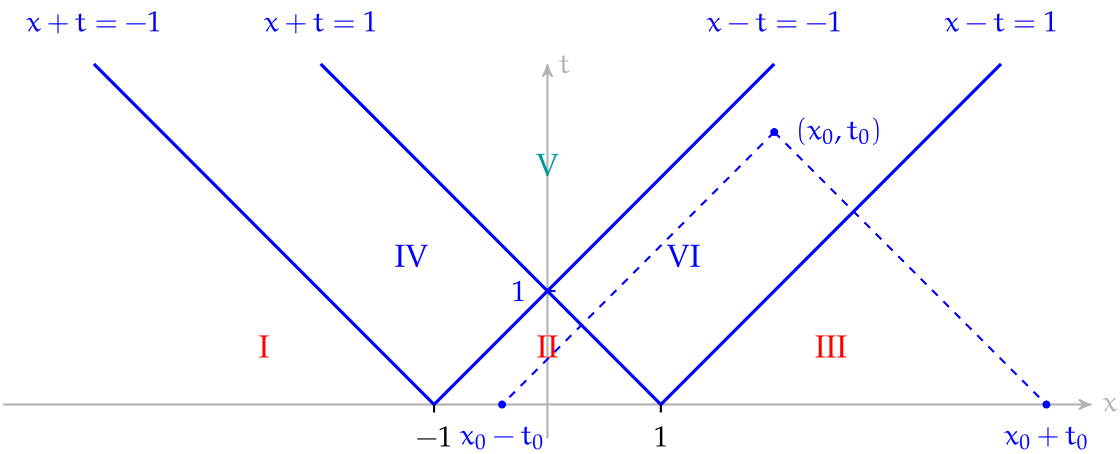
Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} & x \in \mathbb{R} \\ u_t(x,0) = 0. & x \in \mathbb{R} \end{cases}$$

a) Draw the characteristic lines as in the picture at pag. 59 of the Lecture notes for the interval $[a,b] = [-1,1]$. You should have divided the upper-half plane $(x,t) \in \mathbb{R} \times [0,+\infty)$ into the six regions denoted in the picture by I,II,III,IV,V,VI.

Solution:

Here the propagation speed is $c = 1$, so the characteristic lines have the form $\{x \pm t = \text{constant}\}$.



In the picture we drew the required characteristic lines passing through the extremal points of the interval $[-1,1]$, dividing the upper-half plane in the six regions. We also drew an example of how to use this: from any point (x_0,t_0)

we have to draw the characteristic lines from this point and we intersect the x-axis in the two points $x_0 \pm t_0$. The solution in the point (x_0, t_0) is then given by averaging the initial function $f(x)$ in this two points:

$$u(x_0, t_0) = \frac{1}{2}\big(f(x_0 - t_0) + f(x_0 + t_0)\big)$$

In the example we drew in this picture the point (x_0, t_0) belongs to the region VI, which means that the contribution from the point $x_0 + t_0$ is zero because it is outside the interval $[-1, 1]$. In our case this means that for each points in the region VI

$$u(x_0, t_0) = \frac{1}{2} f(x_0 - t_0) = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

b) In this case in each of these regions, and on each of the characteristics line, the solution is equal to a constant. Find all these constants and write them down in the regions and on the lines.

Solution:

We have already partially answered to this question by observing that $u(x, t)$ is constantly equal to $\frac{1}{2}$ in the region VI, and explaining how we obtained this.

The same thing is true in the region IV because we have only the contribute from $x_0 + t_0$.

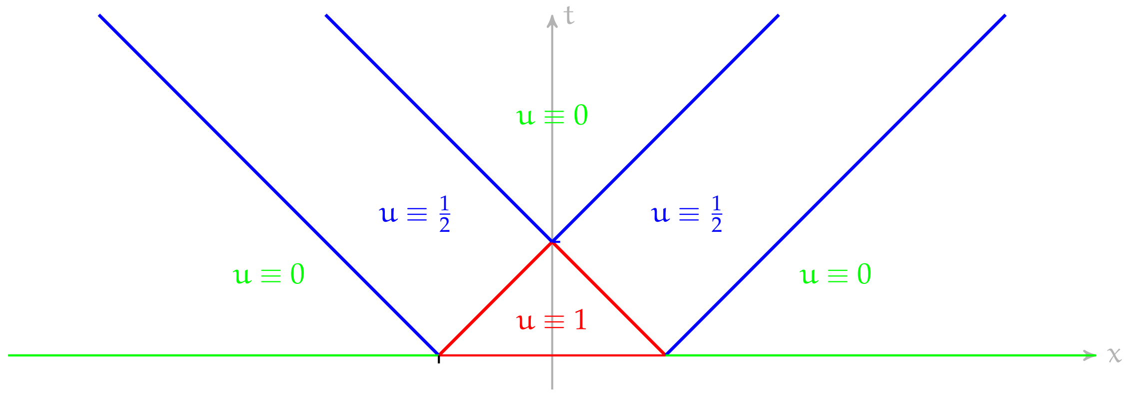
In the regions I,III,V instead the solution is zero because we don't intersect the interval $[-1, 1]$ with the points $x_0 \pm t_0$.

Finally in the region II the solution is equal to 1.

Further discussion is required for the lines in between the regions: as an example along the line $x - t = -1$ the solution is equal to 1 in the segment going from $x = -1$ to $x = 0$ and after it is equal to $\frac{1}{2}$.

We draw in a new graph these results, and in order to avoid confusion we don't report again the labels of the regions and the numbers ± 1 on the axis.

We assign a colour to each of the three possible values of u: red to 1, blue to $\frac{1}{2}$ and green to 0.



c) In this way you have found graphically the solution for each (x,t). What are the maximum and minimum values?

Solution:

From the picture it is clear that:

- $\max_{(x,t)\in\mathbb{R}\times[0,+\infty)} u(x,t) = 1$
- $\min_{(x,t)\in\mathbb{R}\times[0,+\infty)} u(x,t) = 0$

2. Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} & x \in \mathbb{R} \\ u_t(x,0) = g(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0. & |x| > 1 \end{cases} & x \in \mathbb{R} \end{cases}$$

a) Find the values $u(0, \frac{1}{2})$ and $u(\frac{3}{2}, \frac{1}{2})$.

Solution:

We use d'Alembert formula to obtain

$$u(0, \tfrac{1}{2}) = \frac{1}{2}\left(f\left(\tfrac{1}{2}\right) + f\left(-\tfrac{1}{2}\right)\right) + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(s)\,ds = \frac{1}{2}(1+1) + \frac{1}{2}\cdot 1 = \frac{3}{2}.$$

$$u(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{1}{2}\left(f(2) + f(1)\right) + \frac{1}{2}\int_{1}^{2} g(s)\,ds = \frac{1}{2}(0+1) = \frac{1}{2}.$$

b) Find, for each fixed $x \in \mathbb{R}$, the asymptotic limit

$$\lim_{t\to+\infty} u(x,t).$$

Solution:

Let's make a general observation: if $f(x)$ has limits for $x \to \pm\infty$ (call these values $f(\pm\infty)$) and $g(x)$ is integrable on $\mathbb{R}$, then the limit of the solution of the wave equation $u(x,t)$ can be computed as the sum of the three limits:

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} \left(\frac{1}{2}\left(f(x+ct)+f(x-ct)\right) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\,ds \right) =$$

$$= \frac{1}{2}\left(f(+\infty)+f(-\infty)\right) + \frac{1}{2c}\int\limits_{-\infty}^{+\infty} g(s)\,ds.$$

Here $c = 1$, $f(\pm\infty) = 0$ and the integral of $g(x)$ on $\mathbb{R}$ is 2, therefore:

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{2}\left(f(+\infty)+f(-\infty)\right) + \frac{1}{2}\int\limits_{-\infty}^{+\infty} g(s)\,ds = \frac{1}{2}(0+0) + \frac{1}{2}\cdot 2 = 1.$$

Important remark: The asymptotic limit we have computed in the previous exercise $\lim_{t\to+\infty} u(x,t)$ was independent from the $x \in \mathbb{R}$ chosen. Indeed, as said before, this was because f and g were well-behaved, in the sense that f had limits for $x \to \pm\infty$ and g was integrable, so that one can compute the limit of

$$u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\,ds$$

by computing the sum of the limits of all summands. But *pay attention*: this is not always the case!
Easy example in which this doesn't work: just take $c = 1$ and $f(x) = \sin(x)$, while $g(x) = 0$. Then clearly f doesn't have limits at infinity, and $u(x,t)$ will have limit for some x, but not for others:

$$\begin{cases} \lim\limits_{t\to+\infty} u(0,t) = \lim\limits_{t\to+\infty} \frac{1}{2}(\sin(t)-\sin(t)) = \lim\limits_{t\to+\infty} 0 = 0, \\ \lim\limits_{t\to+\infty} u(1,t) = \lim\limits_{t\to+\infty} \frac{1}{2}(\sin(1+t)+\sin(1-t)) = \lim\limits_{t\to+\infty} \sin(1)\cos(t) \quad \rightsquigarrow \quad \text{doesn't exist!} \end{cases}$$

More generally for each x and t

$$u(x,t) = \frac{1}{2}(\sin(x+t)+\sin(x-t)) = \cos(t)\sin(x)$$

and it will have limit for $t \to +\infty$ if and only if $x = k\pi$ with $k \in \mathbb{Z}$.

3. Find, via Fourier series, the solution of the 1-dimensional heat equation with the following initial condition:

$$\begin{cases} u_t = 4\,u_{xx}, & x \in [0,1],\ t \geqslant 0 \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \sin(\pi x) + \sin(5\pi x) + \sin(10\pi x).$$

Use the method of separation of variables from scratch, showing all the steps.

Solution:

With variables separated $u(x,t) = F(x)G(t)$ the differential equation becomes:

$$F(x)\dot{G}(t) = 4F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{4G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{4G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \in [0,+\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \quad \text{and} \quad \dot{G}(t) = 4kG(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

Please turn!

but then imposing the other condition:

$$0 = F(1) = C_1\left(e^{\sqrt{k}} - e^{-\sqrt{k}}\right) \quad \Leftrightarrow \quad \begin{array}{l}\text{either } C_1 = 0\\ \text{or } e^{2\sqrt{k}} = 1\end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(1) = C_1.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

$F(0) = 0$ if and only if $A = 0$. $F(1) = 0$ if and only if $B\sin(p) = 0$, so if we want nontrivial solutions $B \neq 0$, we need to have

$$p = n\pi$$

for some integer $n \geqslant 1$. Conclusion: we have a nontrivial solution for each $n \geqslant 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x)$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -4n^2\pi^2 G$$

which has general solution

$$G_n(t) = C_n e^{-4n^2\pi^2 t}$$

The conclusion is that for every $n \geqslant 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin(n\pi x)e^{-4n^2\pi^2 t}$$

and by the superposition principle:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)e^{-4n^2\pi^2 t}$$

where the coefficients B_n are determined by the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x).$$

Look at the next page!

This case is particularly easy because $f(x)$ is already expressed as a linear combination of these functions and there is no need to compute any integral to get

$$B_n = \begin{cases} 1, & n = 1, 5, 10 \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the solution will be

$$u(x,t) = \sin(\pi x)e^{-4\pi^2 t} + \sin(5\pi x)e^{-100\pi^2 t} + \sin(10\pi x)e^{-400\pi^2 t}$$

4. An aluminium bar of length $L = 1(m)$ has thermal diffusivity of (around)[1]

$$c^2 = 0.0001\left(\frac{m^2}{\sec}\right) = 10^{-4}\left(\frac{m^2}{\sec}\right).$$

It has initial temperature given by $u(x,0) = f(x) = 100\sin(\pi x)$ (°C), and its ends are kept at a constant 0°C temperature. Find the first time t^* for which the whole bar will have temperature $\leqslant 30$°C.
In mathematical terms, solve

$$\begin{cases} u_t = 10^{-4}u_{xx}, \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = 100\sin(\pi x), & 0 \leqslant x \leqslant 1. \end{cases}$$

and find the smallest t^* for which

$$\max_{x\in[0,1]} u(x,t^*) \leqslant 30.$$

You can use the formula from the Lecture notes (pag. 61).

Solution:

The parameters are length $L = 1$, thermal diffusivity $c^2 = 10^{-4}$ and consequently

$$\lambda_n^2 = \frac{c^2 n^2 \pi^2}{L^2} = 10^{-4} n^2 \pi^2.$$

The solution is

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)e^{-\lambda_n^2 t}$$

and

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)$$

[1] we are approximating the standard value which would be $c^2 \approx 0.000097 m^2/\sec$ to make computations easier.

so that the only nontrivial coefficient will be $B_1 = 100$. The solution is explicitely given by

$$u(x,t) = 100\sin(\pi x)e^{-10^{-4}\pi^2 t}.$$

For each fixed time $t \geqslant 0$, it is a multiple of $\sin(\pi x)$, therefore its maximum will be reached in $x = 1/2$ with value

$$M_t := \max_{x\in[0,1]} u(x,t) = u\left(\frac{1}{2}, t\right) = 100\sin\left(\frac{\pi}{2}\right) e^{-10^{-4}\pi^2 t} = 100e^{-10^{-4}\pi^2 t}.$$

This is a decreasing function of t, so that the required value t^* for which the bar will have temperature $\leqslant 30°\mathrm{C}$ is given by imposing

$$M_{t^*} = 30 \quad \Leftrightarrow \quad 100e^{-10^{-4}\pi^2 t^*} = 30 \quad \Leftrightarrow \quad t^* = \frac{10^4}{\pi^2}\ln\left(\frac{10}{3}\right)$$
$$\left(\approx 1219.88 \text{ sec} = 20 \text{ min } 19.88 \text{ sec}\right)$$