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Analysis III

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Solutions Serie 13

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1. Find the solution of the following Laplace equation on the disk of radius 2:

$$\begin{cases} \nabla^2 u = 0, & (x,y) \in D_2 \\ u(x,y) = x^3. & (x,y) \in \partial D_2 \end{cases}$$

Do as follows:

a) Write the boundary condition in polar coordinates.

Solution:

The polar coordinates are linked to the standard cartesian ones by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

At the boundary the radius is $r = 2$ and the function becomes

$$x^3 = (2\cos(\theta))^3 = 8\cos^3(\theta).$$

b) Solve the problem in polar coordinates, using the methods/formulas from the Lecture notes.

[*Hint:* It might be useful at some point the trigonometric formula

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta) \text{]}$$

Solution:

The Poisson integral formula provides always the solution, but it is very difficult to get an explicit solution out of it (solving the integral), so we will try to avoid it whenever possible.
In fact, when the boundary condition is easy, it will be also easy to match the coefficients in the Fourier series.

Namely, from the Lecture notes we have a general solution of the Laplace equation in polar coordinates

$$u(r,\theta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

We impose that on the boundary $r = 2$:

$$u(2,\theta) = \sum_{n=0}^{+\infty} 2^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = 8\cos^3(\theta).$$

Which amounts in finding the Fourier series of $8\cos^3(\theta)$:

$$8\cos^3(\theta) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\theta) + \sum_{n=1}^{+\infty} b_n \sin(n\theta)$$

The formula given in the text of the Exercise allows us to avoid computing any integral to get

$$8\cos^3(\theta) = 6\cos(\theta) + 2\cos(3\theta).$$

In terms of the coefficients we had to find this means

$$\begin{cases} B_n = 0 & \forall n \geqslant 0 \\ A_n = 0 & \forall n \geqslant 0,\, n \neq 1,3 \\ A_1 = \frac{a_1}{2} = 3 \\ A_3 = \frac{a_3}{2^3} = \frac{1}{4} \end{cases}$$

and the solution is

$$u(r,\theta) = rA_1\cos(\theta) + r^3 A_3 \cos(3\theta) = 3r\cos(\theta) + \frac{r^3}{4}\cos(3\theta).$$

c) Express the solution in the standard cartesian coordinates:

$$u(x,y) = ?$$

Solution:

If we want to express some function in terms of the standard cartesian coordinates (x,y) we are ok with terms like $r, \cos^k(\theta), \sin^k(\theta)$. So in this case we just need to write the function $\cos(3\theta)$ in these terms:

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

and

$$\begin{aligned} 3r\cos(\theta) + \frac{r^3}{4}\cos(3\theta) &= 3r\cos(\theta) + \frac{r^3}{4}\left(4\cos^3(\theta) - 3\cos(\theta)\right) = \\ &= 3(r\cos(\theta)) + (r\cos(\theta))^3 - \frac{3}{4}r^2(r\cos(\theta)) = 3x + x^3 - \frac{3}{4}(x^2+y^2)x = \\ &= 3x + x^3 - \frac{3}{4}x^3 - \frac{3}{4}xy^2 = 3x + \frac{1}{4}x^3 - \frac{3}{4}xy^2. \end{aligned}$$

2. **a)** Find the solution $u(r,\theta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \nabla^2 u = 0, & 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant 2\pi \\ u(R,\theta) = \sin^2(\theta). & 0 \leqslant \theta \leqslant 2\pi \end{cases}$$

[*Hint:* Remember the trigonometric formula

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \,]$$

Solution:

The solution will be

$$u(r,\theta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

with coefficients found imposing

$$u(R,\theta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = \sin^2(\theta).$$

Using the trigonometric formula

$$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

we obtain coefficients

$$\begin{cases} B_n = 0, & \forall n \geqslant 0 \\ A_n = 0, & \forall n \geqslant 0, \; n \neq 0, 2 \\ A_0 = \frac{1}{2}, \\ A_2 = -\frac{1}{2R^2}. \end{cases}$$

Finally

$$u(r,\theta) = \frac{1}{2} - \frac{r^2}{2R^2}\cos(2\theta).$$

b) Find the maximum of $u(r,\theta)$. In which point(s) is it reached?

Solution:

The constant $1/2$ plays no role in finding where is the maximum. The other term is a product of functions dependent from different variables. We then just need to maximize these functions independently:

$$\begin{cases} -\cos(2\theta) = 1 \\ r = R \end{cases} \Leftrightarrow \begin{cases} \theta = \frac{\pi}{2}, \frac{3}{2}\pi \\ r = R \end{cases} \rightsquigarrow \quad (r,\theta) = \left(R, \frac{\pi}{2}\right), \left(R, \frac{3}{2}\pi\right) =: P_1, P_2$$

Please turn!

So the maximum is reached in the two points P_1, P_2 and it's equal to

$$u(P_1) = u(P_2) = \frac{1}{2} + \frac{R^2}{2R^2} = \frac{1}{2} + \frac{1}{2} = 1.$$

Remark: the maximum is reached only the boundary. Inside the disk the function is always strictly smaller than this value. You will see in the next lecture that this is not a coincidence.

c) Express the solution in the standard cartesian coordinates.

Solution:

We transform the term

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \rightsquigarrow \quad r^2\cos(2\theta) = x^2 - y^2$$

and the solution will be

$$u(x, y) = \frac{1}{2} - \frac{1}{2R^2}(x^2 - y^2).$$

3. Let u be the unique harmonic function ($\nabla^2 u = 0$) on the unit disk D_1 which on the boundary is

$$u(x, y) = xy + 3, \quad (x, y) \in \partial D_1.$$

Without computing any integral or using any formula from the script, answer the following questions:

a) Find $u(x, y)$.

Solution:

The function on the boundary is well defined everywhere in the plane (x, y). Also, being linear in each variable, it's harmonic:

$$\nabla^2(xy + 3) = \partial_{xx}(xy + 3) + \partial_{yy}(xy + 3) = 0.$$

By uniqueness of the solution for the Dirichlet problem it must be itself the solution:

$$u(x, y) = xy + 3.$$

b) Find the value in the center of the disk: $u(0, 0)$= ?

Solution:

$$u(0, 0) = 0 \cdot 0 + 3 = 3.$$

c) Find the maximum of $u(x, y)$ and in which point(s) it is reached.
Can you notice a similarity with the maximum points in the previous exercise (**2.b**)? What do they have in common?

Solution:

To find the maximum is more convenient to write the solution in polar coordinates in this case

$$u(r, \theta) = r^2 \cos(\theta) \sin(\theta) + 3 = \frac{r^2}{2} \sin(2\theta) + 3.$$

Like we did before we need to impose

$$\begin{cases} \sin(2\theta) = 1 \\ r = R = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta = \frac{\pi}{4}, \frac{5}{4}\pi \\ r = 1 \end{cases} \quad \rightsquigarrow \quad (r, \theta) = \left(1, \frac{\pi}{4}\right), \left(1, \frac{5}{4}\pi\right) =: Q_1, Q_2.$$

So the maximum is reached in the two points Q_1, Q_2 with value

$$u(Q_1) = u(Q_2) = \frac{1^2}{2} 1 + 3 = \frac{1}{2} + 3 = \frac{7}{2}.$$

The similarity with ex. **2.b)** is that the maximum points are on the boundary of the disk. This is because of the maximum principle, which you will see in the next lecture.

4. Prove, without computing explicitely the integrals, that for each $0 \leqslant r < 1$ and for each $0 \leqslant \theta \leqslant 2\pi$:

a)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \, d\phi = 1$$

Solution:

Let $D = \{(r, \theta) \,|\, 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ be the unit disk and u the constant function equal to 1

$$\begin{aligned} u &: D \to \mathbb{R} \\ (r, \theta) &\mapsto u(r, \theta) \equiv 1. \end{aligned}$$

Then clearly u is the solution of the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0, & \text{in } D \\ u(1, \theta) = 1, & 0 \leqslant \theta \leqslant 2\pi \end{cases}$$

and therefore for each $(r, \theta) \in [0,1) \times [0, 2\pi]$ it must be represented by the Poisson integral formula:

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} K(r,\theta,1,\phi) u(1,\phi)\, d\phi \qquad \rightsquigarrow$$

$$\rightsquigarrow \qquad 1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, d\phi.$$

b)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\phi)\sin(\phi) - \sin^3(\phi)\cos(\phi)\big)}{1-2r\cos(\theta-\phi)+r^2}\, d\phi = \frac{r^4}{4}\sin(4\theta).$$

Solution:

We would like to do as we did before. That is, consider the function

$$f(\theta) = \cos^3(\theta)\sin(\theta) - \sin^3(\theta)\cos(\theta), \qquad 0 \leqslant \theta \leqslant 2\pi.$$

We want to solve the Dirichlet problem on the unit disk with this boundary condition

$$\begin{cases} \nabla^2 u = 0, & \text{in } D \\ u(1,\theta) = f(\theta), & 0 \leqslant \theta \leqslant 2\pi \end{cases}$$

We can proceed by matching the coefficients in the Fourier series as did in the Exercises **1.** and **2.**.

Even better, we can also recognise that the boundary condition

$$\cos^3(\theta)\sin(\theta) - \sin^3(\theta)\cos(\theta) = x^3y - xy^3$$

is already a well defined harmonic function on the whole plane

$$\nabla^2(x^3y - xy^3) = 6xy - 6xy \equiv 0.$$

So it must be itself the solution of the Dirichlet problem above.

Now we just need to manipulate it in polar coordinates to obtain

$$x^3y - xy^3 = xy(x^2-y^2) = r^4\cos(\theta)\sin(\theta)\big(\cos^2(\theta) - \sin^2(\theta)\big) =$$
$$= \frac{r^4}{2}\sin(2\theta)\cos(2\theta) = \frac{r^4}{4}\sin(4\theta),$$

and finally using the Poisson integral formula we have proved that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\phi)\sin(\phi) - \sin^3(\phi)\cos(\phi)\big)}{1-2r\cos(\theta-\phi)+r^2}\, d\phi = \frac{r^4}{4}\sin(4\theta).$$