D-MAVT
D-MATL

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Analysis III
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Solutions Serie 14

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1. Let u the unique harmonic function on the disk of radius R which on the boundary is

$$u(x,y) = x^2y^2, \qquad (x,y) \in \partial D_R.$$

Answer, *without finding explicitely the function on the whole disk*, the following questions.

a) Find the value in the center of the disk:

$$u(0,0) = \,?$$

Solution:

We can recover the value of the function in the center of the disk just from the boundary. In fact Poisson's integral formula tells us that for each point (r,θ) inside the disk

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)f(\phi)}{R^2 - 2rR\cos(\theta-\phi)+r^2}\,d\phi$$

In particular the value in the center computes the average of $f(\phi)$ on the boundary

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi.$$

We rewrite our boundary condition in polar coordinates

$$x^2y^2 = (R\cos(\phi))^2(R\sin(\phi))^2 = R^4\cos^2(\phi)\sin^2(\phi),$$

and obtain

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} R^4\cos^2(\phi)\sin^2(\phi)\,d\phi = \frac{R^4}{2\pi}\int_0^{2\pi}\frac{\sin^2(2\phi)}{4}\,d\phi = \frac{R^4}{8\pi}\int_0^{2\pi}\sin^2(2\phi)\,d\phi.$$

This can be either computed with usual integration by parts or using this *Alternative method:* observe that

$$\sin\left(x+\frac{\pi}{2}\right) = -\cos(x) \qquad \rightsquigarrow \qquad \sin^2\left(x+\frac{\pi}{2}\right) = \cos^2(x). \tag{1}$$

But then changing variables $\psi = \phi - \pi/4$, and using the 2π-periodicity of all these functions we get

$$\int_0^{2\pi} \sin^2(2\phi)\,d\phi \overset{\substack{\text{change}\\\text{of}\\\text{variables}}}{=} \int_{-\frac{\pi}{4}}^{2\pi-\frac{\pi}{4}} \sin^2\left(2\psi+\frac{\pi}{2}\right)d\psi \overset{(1)}{=} \int_{-\frac{\pi}{4}}^{2\pi-\frac{\pi}{4}} \cos^2(2\psi)\,d\psi \overset{\substack{2\pi\text{-}\\\text{periodicity}}}{=}$$

$$= \int_0^{2\pi} \cos^2(2\psi)\,d\psi.$$

Finally observe that the sum of these two (equal) integral is

$$\int_0^{2\pi} \sin^2(2\phi)\,d\phi + \int_0^{2\pi} \cos^2(2\phi)\,d\phi = \int_0^{2\pi} \left(\sin^2(2\phi)+\cos^2(2\phi)\right)d\phi = \int_0^{2\pi} 1\,d\phi = 2\pi$$

and so

$$\rightsquigarrow \qquad \int_0^{2\pi} \sin^2(2\phi)\,d\phi = \int_0^{2\pi} \cos^2(2\phi)\,d\phi = \pi.$$

Coming back to the original question of the exericse

$$u(0,0) = \frac{R^4}{8\pi}\int_0^{2\pi} \sin^2(2\phi)\,d\phi = \frac{R^4}{8\cancel{\pi}}\cdot\cancel{\pi} = \frac{R^4}{8}.$$

b) Find the maximum of u on the disk:

$$\max_{(x,y)\in D_R} u(x,y) = ?$$

Solution:

By the maximum principle we know that the maximum is assumed on the boundary

$$\max_{(x,y)\in D_R} u(x,y) = \max_{(x,y)\in \partial D_R} u(x,y) = \max_{\theta\in[0,2\pi]} u(R,\theta) = \max_{\theta\in[0,2\pi]} \frac{R^4}{4}\sin^2(2\theta) = \frac{R^4}{4}$$

Look at the next page!

c) Same question for the minimum.

Solution:

As the maximum principle, there is also a minimum principle. Observe that a function u solves the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = f & \text{on } \partial D_R \end{cases}$$

if and only if the function $v := -u$ solves the Dirichlet problem

$$\begin{cases} \nabla^2 v = 0, & \text{in } D_R \\ v = -f. & \text{on } \partial D_R \end{cases}$$

Therefore, using the maximum principle for v, and the observation that for any function h its maximum and minimum are linked by

$$\max h = -\min(-h) \quad \& \quad \min h = -\max(-h),$$

we get the minimum principle: also the minimum of a harmonic function is reached on the boundary.

$$\min_{D_R} u = -\max_{D_R}(-u) = -\max_{D_R} v = -\max_{\partial D_R} v = -\max_{\partial D_R}(-u) = \min_{\partial D_R} u.$$

So we need to find the minimum

$$\min_{D_R} u = \min_{\partial D_R} u = \min_{\theta \in [0,2\pi]} u(R, \theta) = \min_{\theta \in [0,2\pi]} \frac{R^4}{4} \sin^2(2\theta) = 0.$$

2. Answer the following questions.

a) Let $a, b \in \mathbb{N}(= \{0, 1, 2, \dots\})$ and u the solution of the following Laplace equation:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u = x^a y^b, & \partial D_R \end{cases}$$

For which values of a, b is it true that $u(0, 0) = 0$?

Hint: You should find that the answer depends on their parity.

Solution:

The value in the center is determined by the average of the boundary function. In polar coordinates our function is

$$x^a y^b = (R\cos(\theta))^a (R\sin(\theta))^b = R^{a+b} \cos^a(\theta) \sin^b(\theta).$$

Therefore the question becomes when:

$$\frac{1}{2\pi}\int_0^{2\pi} R^{a+b}\cos^a(\theta)\sin^b(\theta)\,d\theta = 0.$$

Let's call, for each integers $a, b \geqslant 0$, the relevant part of this integral by $I_{a,b}$ and observe that, by 2π-periodicity, it is also equal to

$$I_{a,b} := \int_0^{2\pi} \cos^a(\theta)\sin^b(\theta)\,d\theta = \int_{-\pi}^{\pi} \cos^a(\theta)\sin^b(\theta)\,d\theta.$$

Observe that

$$\begin{cases} \cos(\theta) \text{ even} \\ \sin(\theta) \text{ odd} \end{cases} \quad \rightsquigarrow \quad \begin{cases} \cos^a(\theta) \ \text{even}, & \forall a \\ \sin^b(\theta) \ \begin{cases} \text{even}, & b \text{ even} \\ \text{odd}, & b \text{ odd} \end{cases} \end{cases}$$

We can conclude that for b is odd, and any a, we integrate an odd function on $[-\pi, \pi]$ and therefore $I_{a,b} = 0$. We need to analyse the case $b = 2k$ even, for which the product is even and we have

$$I_{a,2k} = \int_{-\pi}^{\pi} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta = 2\int_0^{\pi} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta.$$

If we split the interval $[0, \pi]$ in half we can notice one last simmetry, the one with respect to the axis $x = \pi/2$, and splitting the integral in these two parts we get

$$\begin{cases} \cos(\pi - x) = -\cos(x) \\ \sin(\pi - x) = \sin(x) \end{cases} \quad \rightsquigarrow \quad \int_0^{\pi} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\pi} \underbrace{\cos^a(\theta)\sin^{2k}(\theta)\,d\theta}_{\text{change of variables } \theta = \pi - x}$$

$$= \int_0^{\frac{\pi}{2}} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta + (-1)^a \int_0^{\frac{\pi}{2}} \cos^a(x)\sin^{2k}(x)\,dx =$$

$$= \begin{cases} 2\int_0^{\frac{\pi}{2}} \cos^a(\theta)\sin^{2k}(\theta)\,d\theta > 0, & a \ \text{even} \\ 0, & a \ \text{odd}. \end{cases}$$

The reason why the first integral is greater than zero is because the integrand is a positive function on that interval.
The conclusion is

$$u(0,0) = 0 \quad \Leftrightarrow \quad I_{a,b} = 0 \quad \Leftrightarrow \quad \begin{cases} b \ \text{odd}, \ \forall a \\ b \ \text{even}, \ a \ \text{odd}. \end{cases}$$

Look at the next page!

b) Let u be the solution of the following Laplace equation:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(R,\theta) = 3R\, e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}. & \partial D_R \end{cases}$$

Find the only radius R for which

$$\max_{(x,y)\in D_R} u(x,y) = \pi$$

Solution:

By the maximum principle the maximum is reached on the boundary

$$\max_{D_R} u = \max_{\partial D_R} u = \max_{\theta\in[0,2\pi]} u(R,\theta) = 3R \max_{\theta\in[0,2\pi]} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}.$$

The exponential is a strictly increasing function so we just need to find the maximum of the argument, that is

$$\max_{\theta\in[0,2\pi]} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}} = e^{\max_{\theta\in[0,2\pi]} \frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}}$$

Analysing this rational function

$$g(\theta) = \frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}, \quad \theta \in [0,2\pi]$$

we notice that the limit approaching 0 from the right and 2π from the left is $-\infty$, and that it is always strictly negative, apart from $\theta = \pi$, in which it's zero. Therefore

$$\max_{D_R} u = 3R \max_{\theta\in[0,2\pi]} e^{\frac{(\theta-\pi)^2}{\theta(\theta-2\pi)}} = 3Re^{\max_{\theta\in[0,2\pi]} \frac{(\theta-\pi]^2}{\theta(\theta-2\pi)}} = 3Re^0 = 3R.$$

The conclusion is

$$\max_{(x,y)\in D_R} u(x,y) = \pi \qquad \Leftrightarrow \qquad 3R = \pi \qquad \Leftrightarrow \qquad R = \frac{\pi}{3}$$

c) Let u be the solution of:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(R,\theta) = \sin^9(\theta). & \partial D_R \end{cases}$$

Is it true or false that $u + 1 \geqslant 0$ everywhere on the disk?.

Solution:

By the minimum principle (explained above), the minimum will be on the boundary, therefore

$$\min_{D_R} u = \min_{\theta\in[0,2\pi]} \sin^9(\theta) = -1$$

and the answer is yes, $u + 1 \geqslant 0$ everywhere on the disk.

3. Poisson's equation on a disk

Solve the following Poisson's equation on a disk of radius R:

$$\begin{cases} \nabla^2 u = x^2 + y^2, & D_R \\ u = 0, & \partial D_R \end{cases}$$

Proceed as follows.

a) Find a function g such that $\nabla^2 g = x^2 + y^2$.

<u>*Solution:*</u>

It's convenient first to write the problem in polar coordinates:

$$\nabla^2 g = g_{rr} + \frac{1}{r^2} g_{\theta\theta} + \frac{1}{r} g_r = r^2.$$

Because the function r^2 depends only on r we first look for functions $g = g(r)$, therefore for solutions of:

$$g_{rr} + \frac{1}{r} g_r = r^2.$$

Searching for solutions of the form $c_\alpha r^\alpha$ (this Ansatz is justified by the specific form of this differential equation and its coeffcients) one finds the solution

$$g(r) = \frac{1}{16} r^4.$$

b) Observe that the function $v := u - g$ solves the Laplace equation on the disk with some nonzero boundary condition. Write down this Dirichlet problem for v.

<u>*Solution:*</u>

The Dirichlet problem that $v = u - g$ satisfies is

$$\begin{cases} \nabla^2 v = 0, & D_R \\ v = -\frac{1}{16} R^4, & \partial D_R \end{cases}$$

c) Solve this problem for v and then u will be $u = v + g$.

<u>*Solution:*</u>

The solution of this problem is obvious because the boundary function is constant, therefore this constant solves the problem itself

$$v = -\frac{1}{16} R^4.$$

As a consequence we found the solution

$$\boxed{u = v + g = -\frac{1}{16} R^4 + \frac{1}{16} r^4 = \frac{1}{16}\left(r^4 - R^4\right) = \frac{1}{16}\left(x^4 + 2x^2y^2 + y^4 - R^4\right).}$$

Look at the next page!

More generally: Generalising what has been done here, we can say that the solution of the following Poisson's equation

$$\begin{cases} \nabla^2 u = f, & D_R \\ u = 0, & \partial D_R \end{cases}$$

where $f = f(r)$ is a function of the radius, is given by

$$u = g - g(R),$$

where $g = g(r)$ is a solution of

$$\nabla^2 g = g_{rr} + \frac{1}{r} g_r = f.$$