D-MAVT
D-MATL

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Autumn 2019

Analysis III

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Solutions Serie 2

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1. Find the Laplace transform $F(s) := \mathcal{L}(f)(s)$ of the following functions:

a) $f(t) = t^2 + 4t + 1$

Solution:

From the lecture we know that the polynomial $g_n(t) = t^n$ ($n \in \mathbb{N}$), has Laplace transform, defined for each $s > 0$: $G_n(s) = \frac{n!}{s^{n+1}}$. Using the linearity of $\mathcal{L}$ we have

$$F(s) = \mathscr{L}(t^2 + 4t + 1)(s) = \mathscr{L}(t^2)(s) + 4\mathscr{L}(t)(s) + \mathscr{L}(1)(s) = \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{1}{s} = \\ = \frac{s^2 + 4s + 2}{s^3}.$$

b) $f(t) = \dfrac{1}{\sqrt{t}}$, using that

$$\Gamma\left(\frac{1}{2}\right) \left(= \int_0^{+\infty} t^{-1/2} e^{-t} dt \right) = \sqrt{\pi}$$

Solution:

For each $s > 0$, we make the change of variables $u = st$, so that $dt = 1/s\, du$, and

$$F(s) = \int_0^{+\infty} t^{-1/2} e^{-st} dt = \frac{1}{s} \int_0^{+\infty} s^{1/2} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

c) $f(t) = \sin(\omega t), \quad \omega \in \mathbb{R}$

Solution 1:

The Laplace transform can be computed directly by definition, integrating by parts

$$\mathscr{L}(\sin(\omega t))(s) = \int_0^{+\infty} e^{-st}\sin(\omega t)\,dt = \cancel{-\frac{1}{s}e^{-st}\sin(\omega t)\Big|_0^{+\infty}} + \frac{\omega}{s}\int_0^{+\infty} e^{-st}\cos(\omega t)\,dt =$$

$$= -\frac{\omega}{s^2}e^{-st}\cos(\omega t)\Big|_0^{+\infty} - \frac{\omega^2}{s^2}\int_0^{+\infty} e^{-st}\sin(\omega t)\,dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathscr{L}(\sin(\omega t))(s)$$

$$\iff \quad \mathscr{L}(\sin(\omega t))(s) = \frac{\omega}{s^2+\omega^2}.$$

Solution 2 (do not try this at home):

The Laplace transform has been defined just for real-valued functions. It can be defined in the same way for complex-valued functions, and also the variable s is assumed to be complex. Of course in the particular case of real-valued functions we have our old definition. From the lecture we know the Laplace transform of the real-valued exponential e^{at}, $a \in \mathbb{R}$, but also for the imaginary exponential $e^{i\omega t}$, $\omega \in \mathbb{R}$, the same computation makes sense and

$$\mathcal{L}\left(e^{i\omega t}\right)(s) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}, \qquad \mathfrak{Re}(s) > 0$$

But $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, thus by linearity

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i\,\mathcal{L}(\sin(\omega t)).$$

Finally comparing real and imaginary parts of the previous two equations we get

$$\mathcal{L}(\cos(\omega t))(s) = \frac{s}{s^2+\omega^2}, \qquad \mathcal{L}(\sin(\omega t))(s) = \frac{\omega}{s^2+\omega^2}.$$

d) $f(t) = \cos(\omega t), \quad \omega \in \mathbb{R}$

Solution 1:

We started computing the Laplace transform of $\sin(\omega t)$ with

$$\mathscr{L}(\sin(\omega t))(s) = \int_0^{+\infty} e^{-st}\sin(\omega t)\,dt = \frac{\omega}{s}\int_0^{+\infty} e^{-st}\cos(\omega t)\,dt = \frac{\omega}{s}\mathcal{L}(\cos(\omega t))(s),$$

from which we get:

$$\mathcal{L}(\cos(\omega t))(s) = \frac{s}{s^2+\omega^2}.$$

Solution 2:

It's the strategy used in the Solution 2 of Exercise **c)**, from which we obtained the Laplace transforms of both sine and cosine.

e) $f(t) = \sin(\alpha t)\cos(\beta t), \quad \alpha, \beta \in \mathbb{R}$

Solution:

From Exercise **3.b)** of Serie 1 we know that

$$\sin(\alpha t)\cos(\beta t) = \frac{1}{2}\left(\sin((\alpha+\beta)t) + \sin((\alpha-\beta)t)\right).$$

In Exercise **c)** we computed the Laplace transform of the sine. Thus

$$\mathcal{L}(\sin(\alpha t)\cos(\beta t))(s) = \frac{1}{2}\left(\frac{\alpha+\beta}{s^2+(\alpha+\beta)^2} + \frac{\alpha-\beta}{s^2+(\alpha-\beta)^2}\right).$$

2. Given a function $f(t)$ denote its Laplace transform by $F(s) := \mathcal{L}(f(t))(s)$.

a) If not already done in the lecture, prove that for functions $f(t)$ for which both their Laplace transform and the Laplace transform of $tf(t)$ is well-defined:

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}F(s). \tag{1}$$

Solution:

Recall the definition of the Laplace transform:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^{+\infty} e^{-st} f(t)\, dt.$$

The assumptions that both this, and the Laplace transform of $tf(t)$ exist, allow us to differentiate under the integral sign[1]. Therefore what we get is straightforward:

$$-\frac{d}{ds}F(s) = -\int_0^{+\infty} \frac{d}{ds}\left(e^{-st}f(t)\right)\, dt = \int_0^{+\infty} e^{-st} t f(t)\, dt = \mathcal{L}(tf(t))(s).$$

[1] **Theory:** The general principle underlying all operations of exchanging limits with integrals is the Lebesgue's dominated convergence theorem. The derivative is a particular case of limit.

b) Prove that for functions $f(t)$ for which both their Laplace transform and the Laplace transforms of $t^i f(t)$ (for all indices $i = 0, 1, \dots, n$) are well-defined:

$$\mathcal{L}(t^n f(t))(s) = (-1)^n \frac{d^n}{ds^n} F(s). \tag{2}$$

Solution:

We can simply iterate several times the previous result and obtain:

$$\mathcal{L}(t^n f(t))(s) = \mathcal{L}(t \cdot t^{n-1} f(t))(s) \overset{\textbf{a)}}{=} -\frac{d}{ds}\mathcal{L}(t^{n-1} f(t))(s) \overset{\textbf{a)}}{=}$$
$$\overset{\textbf{a)}}{=} (-1)^2 \frac{d^2}{ds^2}\mathcal{L}(t^{n-2} f(t))(s) = \dots = (-1)^n \frac{d^n}{ds^n} F(s).$$

c) Choose $f(t) = 1$ and use **b)** to obtain another proof of

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

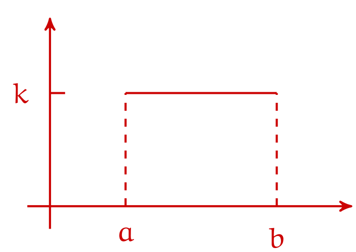
Solution:

In particular for $f(t) = 1$ we have $\mathcal{L}(f)(s) = 1/s$ and then

$$\frac{d^n}{ds^n}\left(\frac{1}{s}\right) = (-1)^n \frac{n!}{s^{n+1}} \implies \mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}.$$

3. Find the Laplace transform of the following functions:

a)



Solution:

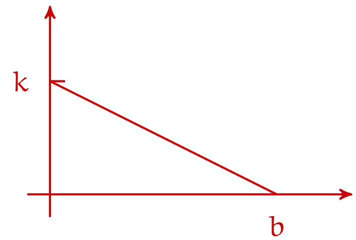
We have

$$f(t) = \begin{cases} k, & a \leqslant t \leqslant b \\ 0, & \text{otherwise} \end{cases}$$

and then

$$\mathscr{L}(f)(s) = k\int_a^b e^{-st}\,dt = -\frac{k}{s}e^{-st}\Big|_a^b = \frac{k}{s}(e^{-sa} - e^{-sb}).$$

b)



Solution:

We have

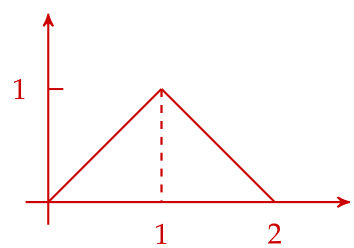
$$f(t) = \begin{cases} k(1-\frac{t}{b}), & 0 \leqslant t \leqslant b \\ 0, & \text{otherwise.} \end{cases}$$

Then, integrating by parts

$$\mathscr{L}(f)(s) = k\int_0^b e^{-st}\left(1-\frac{t}{b}\right)\,dt = -\frac{k}{s}e^{-st}\left(1-\frac{t}{b}\right)\Big|_0^b - \frac{k}{bs}\int_0^b e^{-st}\,dt =$$

$$= \frac{k}{s} + \frac{k}{bs^2}e^{-st}\Big|_0^b = \frac{k}{s} + \frac{k}{bs^2}e^{-sb} - \frac{k}{bs^2} = \frac{k}{bs^2}\left(bs + e^{-sb} - 1\right).$$

c)



Solution:

We have

$$f(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2-t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Again, integrating by parts

$$\mathscr{L}(f)(s) = \int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt = -\frac{t}{s}e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{2-t}{s}e^{-st}\Big|_1^2 - \frac{1}{s}\int_1^2 e^{-st}\,dt =$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-st}\Big|_0^1 + \frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-st}\Big|_1^2 = -\frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \frac{1}{s^2}e^{-2s} - \frac{1}{s^2}e^{-s} =$$

$$= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{(1-e^{-s})^2}{s^2}.$$

Please turn!

4. Find the <u>inverse</u> Laplace transform $f = \mathscr{L}^{-1}(F)$ of the following functions:

a) $F(s) = \dfrac{1}{s^4}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{1}{s^4}\right) = \mathscr{L}^{-1}\left(\frac{1}{3!}\cdot\frac{3!}{s^4}\right) = \frac{1}{3!}\cdot\mathscr{L}^{-1}\left(\frac{3!}{s^4}\right) = \frac{1}{3!}t^3.$$

b) $F(s) = \dfrac{1}{(s-8)^{10}}$

Solution:

Recall the s-shifting property, for which

$$\mathcal{L}(e^{at}f(t))(s) = \mathcal{L}(f)(s-a), \quad a \in \mathbb{R}.$$

Therefore:

$$\mathscr{L}^{-1}\left(\frac{1}{(s-8)^{10}}\right) = \mathscr{L}^{-1}\left(\frac{1}{9!}\cdot\frac{9!}{(s-8)^{10}}\right) = \frac{1}{9!}\cdot\mathscr{L}^{-1}\left(\frac{9!}{(s-8)^{10}}\right) = \frac{1}{9!}e^{8t}t^9.$$

c) $F(s) = \dfrac{s+3}{s^2-9}$

Solution:

$$\frac{s+3}{s^2-9} = \frac{1}{s-3} \implies \mathscr{L}^{-1}\left(\frac{s+3}{s^2-9}\right) = \mathscr{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}.$$

d) $F(s) = \dfrac{24}{(s-5)(s+3)}$

Solution:

By partial fraction decomposition

$$\frac{24}{(s-5)(s+3)} = \frac{24}{8}\left(\frac{1}{s-5} - \frac{1}{s+3}\right) = 3\left(\frac{1}{s-5} - \frac{1}{s+3}\right) \implies f(t) = 3(e^{5t} - e^{-3t}).$$

e) $F(s) = \dfrac{1}{s^2+4}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right) = \mathscr{L}^{-1}\left(\frac{1}{2}\cdot\frac{2}{s^2+4}\right) = \frac{1}{2}\cdot\mathscr{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin(2t).$$

f) $F(s) = \dfrac{1}{s^2 + 4s + 20}$

Solution:

We would like to write $s^2 + 4s + 20$ in the form $(s + a)^2 + \omega^2$. In fact this is possible for $a = 2, \omega = 4$. After we can use the s-shifting property and the Laplace transform of the sine to get:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2 + 4s + 20}\right) = \mathscr{L}^{-1}\left(\frac{1}{(s+2)^2 + 4^2}\right) = \frac{1}{4}e^{-2t}\sin(4t).$$

g) $F(s) = \dfrac{s + 1}{(s + 2)(s^2 + s + 1)}$

Solution:

The strategy is first applying partial fraction decomposition and then trying to get some expression similar to the Laplace transforms of sine and cosine.

$$F(s) = \frac{s+1}{(s+2)(s^2+s+1)} = \frac{1}{3}\left(\frac{s+2}{s^2+s+1} - \frac{1}{s+2}\right) = \frac{1}{3}\left(\frac{s+2}{s^2+s+1}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

The second term is the Laplace transform of $-1/3e^{-2t}$ so we just have to modify further the first term.

$$\frac{s+2}{s^2+s+1} = \frac{s+2}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{s+2}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \sqrt{3}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \Longrightarrow$$

$$\Longrightarrow \quad \mathscr{L}^{-1}\left(\frac{s+2}{s^2+s+1}\right) = e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Finally the whole inverse Laplace transform will be

$$f = \mathscr{L}^{-1}\left(\frac{s+1}{(s+2)(s^2+s+1)}\right) = \frac{1}{3}e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}e^{-2t}.$$

h) $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$

Solution:

We still use first partial fraction decomposition to get terms which are known Laplace transforms (exponential, polynomials, sine and cosine).

$$\frac{s}{(s-1)^2(s^2+2s+5)} = \frac{1}{16}\cdot\frac{1}{s-1} + \frac{1}{8}\cdot\frac{1}{(s-1)^2} - \frac{1}{16}\cdot\frac{s}{s^2+2s+5} - \frac{5}{16}\cdot\frac{1}{s^2+2s+5} =$$
$$= \frac{1}{16}\cdot\frac{1}{s-1} + \frac{1}{8}\cdot\frac{1}{(s-1)^2} - \frac{1}{16}\cdot\frac{s+1-1}{(s+1)^2+2^2} - \frac{5}{16}\cdot\frac{1}{(s+1)^2+2^2} =$$
$$= \frac{1}{16}\cdot\frac{1}{s-1} + \frac{1}{8}\cdot\frac{1}{(s-1)^2} - \frac{1}{16}\cdot\frac{s+1}{(s+1)^2+2^2} - \frac{4}{16}\cdot\frac{1}{(s+1)^2+2^2} \implies$$
$$\implies \quad f = \mathscr{L}^{-1}\left(\frac{s}{(s-1)^2(s^2+2s+5)}\right) = \frac{e^t}{16} + \frac{te^t}{8} - \frac{1}{16}e^{-t}\left(\cos(2t) + 2\sin(2t)\right).$$

5. (Bonus exercise)

a) (For those who have never seen this)
Exercise **1.b)** assumes that $\Gamma(1/2) = \sqrt{\pi}$; this exercise proves it.
Let us call $I := \Gamma(1/2)$ this value:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}\,dt.$$

(i) Use an opportune change of variables to prove that:

$$I = 2\int_0^{+\infty} e^{-x^2}dx.$$

Solution:
We change variables $t = x^2$, for which $dt = 2xdx$, and:

$$I = \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}dt \overset{t=x^2}{=} \int_0^{+\infty} x^{-1}e^{-x^2}2xdx = 2\int_0^{+\infty} e^{-x^2}dx.$$

(ii) Justify why

$$2\int_0^{+\infty} e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{-x^2}dx.$$

Solution:
The function in the integral is symmetric with respect to $x = 0$. Thus the integral over all real numbers is twice the integral from 0 to $+\infty$.

(iii) Compute this integral in a smart way by computing its square. Fill the dots (with some polar coordinates) to get

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy = \\ = \cdots = \pi$$

Solution:
We fill the dots using polar coordinates[2], for which $dxdy = r dr d\vartheta$:

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy = \\ = \int_0^{+\infty}\int_0^{2\pi} e^{-r^2} r dr d\vartheta = 2\pi \int_0^{+\infty} e^{-r^2} r dr = 2\pi \cdot \left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{+\infty} = \pi.$$

(iv) From (iii) we can deduce that the desired value is one of the two square roots of π: $I = \pm\sqrt{\pi}$.
Why can we exclude the negative value?
Solution:
Because the function in the integral is $\geqslant 0$, therefore also its integral must be.

Theory reminder for Ex 5.b): The Laplace transform of a finite linear combination of functions is the linear combination of their Laplace transforms:

$$\mathcal{L}(a_1 f_1 + \cdots + a_m f_m) = a_1\mathcal{L}(f_1) + \cdots + a_m\mathcal{L}(f_m), \quad a_1, \ldots, a_m \in \mathbb{R}.$$

The same thing is true for infinite linear combination of functions under opportune conditions of convergence which are all satisfied in the following cases. To explain better, let's pretend we don't know the Laplace transform of the exponential and let's compute it explicitely from its power series expression

$$\mathcal{L}(e^{at})(s) = \mathcal{L}\left(\sum_{k=0}^{+\infty}\frac{(at)^k}{k!}\right)(s) = \sum_{k=0}^{+\infty}\frac{a^k}{k!}\mathcal{L}(t^k)(s) = \\ = \sum_{k=0}^{+\infty}\frac{a^k}{\cancel{k!}}\cdot\frac{\cancel{k!}}{s^{k+1}} = \frac{1}{s}\sum_{k=0}^{+\infty}\left(\frac{a}{s}\right)^k = \frac{1}{s}\cdot\frac{1}{1-\frac{a}{s}} = \frac{1}{s-a}.$$

[2]Remember: $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$.

b) (i) Compute again $\mathcal{L}(\sin(\omega t))(s)$ using this method. Remember that:

$$\sin(\omega t) = \sum_{k=0}^{+\infty} (-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}$$

Solution:
We have

$$\mathcal{L}(\sin(\omega t))(s) = \mathcal{L}\left(\sum_{k=0}^{+\infty} (-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}\right)(s) = \sum_{k=0}^{+\infty} (-1)^k \frac{\omega^{2k+1}}{(2k+1)!} \mathcal{L}(t^{2k+1})(s) =$$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{\omega^{2k+1}}{\cancel{(2k+1)!}} \cdot \frac{\cancel{(2k+1)!}}{s^{2k+2}} = \frac{\omega}{s^2} \sum_{k=0}^{+\infty} \left(-\frac{\omega^2}{s^2}\right)^k = \frac{\omega}{s^2} \cdot \frac{1}{1+\frac{\omega^2}{s^2}} = \frac{\omega}{s^2+\omega^2}.$$

(ii) Prove that:

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \arctan\left(\frac{1}{s}\right).$$

Remember that the power series expansion of the arctangent is:

$$\arctan(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)}.$$

Solution:

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right)(s) = \mathcal{L}\left(\sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k}}{(2k+1)!}\right)(s) = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{(2k+1)!} \mathcal{L}(t^{2k})(s) =$$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{1}{(2k+1)!} \cdot \frac{(2k)!}{s^{2k+1}} = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{(2k+1)} \cdot \frac{1}{s^{2k+1}} = \arctan\left(\frac{1}{s}\right).$$