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Analysis III

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Solutions Serie 6

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1. Consider the function

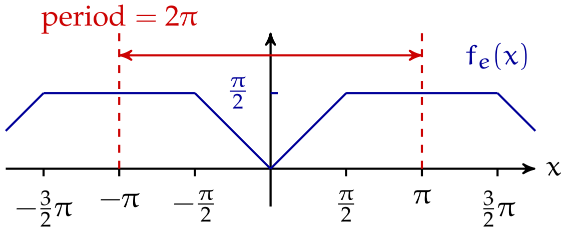
$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi \end{cases}$$

a) Extend f to an even function on the interval $[-\pi, \pi]$ and then finally to an even, 2π-periodic function on $\mathbb{R}$ and call this function f_e.
Sketch the graph of f_e and find its Fourier series.

Solution:

The even extension f_e is given, in the interval $[-\pi, \pi]$, by

$$f_e(x) = \begin{cases} \frac{\pi}{2}, & -\pi \leqslant x \leqslant -\frac{\pi}{2} \\ -x, & -\frac{\pi}{2} \leqslant x \leqslant 0 \\ x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}$$



Being even, the b_n coefficients will vanish, while

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \tfrac{\pi}{2}\,dx = \frac{1}{2\pi}x^2\Big|_0^{\frac{\pi}{2}} + \frac{x}{2}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{3\pi}{8},$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos(nx)\,dx = \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos(nx)\,dx = \\
&= \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{1}{n}\int_0^{\frac{\pi}{2}}\sin(nx)\,dx\right) + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} = \\
&= \frac{1}{n}\sin\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} = \\
&= \frac{2}{n^2\pi}\left(\cos\left(\tfrac{n\pi}{2}\right) - 1\right) = \begin{cases} -\frac{2}{n^2\pi}, & n = 2j+1 \\ \frac{2}{n^2\pi}((-1)^j - 1), & n = 2j. \end{cases}
\end{aligned}
$$

The Fourier series is thus

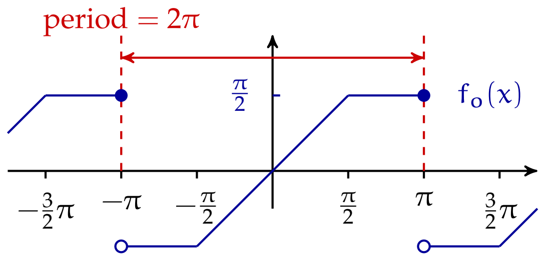
$$
\frac{3\pi}{8} + \frac{2}{\pi}\sum_{j=1}^{+\infty}\frac{1}{(2j)^2}((-1)^j - 1)\cos(2jx) - \frac{2}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^2}\cos((2j+1)x).
$$

b) Do the same for the odd, 2π-periodic extension[1] of f (call this f_o).

Solution:

The odd extension f_o is given, in the interval $(-\pi, \pi]$, by

$$
f_o(x) = \begin{cases} -\frac{\pi}{2}, & -\pi < x \leqslant -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \leqslant x \leqslant 0 \\ x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}
$$



[1] To be precise, we can't extend f to an odd, periodic function everywhere because $f(\pi) = \pi$ is not zero. The problematic points are $\pi + 2k\pi, k \in \mathbb{Z}$. Let's assign to these points the value π.

Look at the next page!

Therefore here the a_n coefficients will be all zero, while

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f_o(x)\sin(nx)\,dx =$$
$$= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin(nx)\,dx$$
$$= \frac{2}{\pi}\left(-\frac{x}{n}\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(nx)\,dx\right) - \frac{1}{n}\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi}$$
$$= -\frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right)$$
$$= \frac{2}{n^2\pi}\sin\left(\tfrac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi) = \begin{cases} -\frac{1}{n}, & n = 2j \\ \frac{2}{n^2\pi}(-1)^j + \frac{1}{n}, & n = 2j+1 \end{cases}$$

and the Fourier series is

$$-\sum_{j=1}^{+\infty}\frac{1}{2j}\sin(2jx) + \sum_{j=0}^{+\infty}\left(\frac{2}{(2j+1)^2\pi}(-1)^j + \frac{1}{2j+1}\right)\sin((2j+1)x).$$

2. Find the Fourier series of the 2L-periodic extension of

$$f(x) = x, \quad x \in [-L, L)$$

considered in Exercise **5.a)** of Serie 5.

Solution:

The extended function is odd[2] and therefore all a_n coefficients are going to vanish. Integration by parts (and a change of variable $y = \pi x/L$) yields

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{L}\cdot\frac{L^2}{\pi^2}\int_{-\pi}^{\pi} y\sin(ny)\,dy =$$
$$= \frac{L}{\pi^2}\cdot\frac{-ny\cos(ny)+\sin(ny)}{n^2}\Bigg|_{-\pi}^{\pi} =$$
$$= -\frac{L}{\pi^2 n^2}\cdot\left(n\pi(-1)^n - n(-\pi)(-1)^n\right) = -\frac{2L}{\pi n}(-1)^n.$$

[2] To be precise, it is odd almost everywhere, in the sense that $f(-x) = -f(x)$ for all x apart from a discrete set. In fact when we extend it to a 2L periodic function we get for example $f(L) = -L = f(-L)$. Anyway, because we are integrating the function, these points don't contribute.

Therefore the Fourier series is

$$\sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{2L}{\pi n} \sin\left(\frac{n\pi}{L}x\right).$$

3. Find the complex Fourier series of the same function $f(x)$ considered in the previous exercise. Verify that the coefficients c_n of this series

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

are related as written in the script to the real coefficients a_n, b_n found in the previous exercise.
If you have not computed it before: the real Fourier series of f is

$$\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{2L}{\pi n} \sin\left(\frac{n\pi}{L}x\right) \quad \rightsquigarrow \quad \begin{cases} a_n = 0 \\ b_n = (-1)^{n+1}\frac{2L}{\pi n} \end{cases}$$

Solution:

The complex Fourier coefficients for f are, for $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} x e^{-i\frac{n\pi}{L}x}\,dx = \frac{L}{2\pi^2}\int_{-\pi}^{\pi} y e^{-iny}\,dy = \\ &= \frac{L}{2\pi^2}\left(-\frac{y}{in}e^{-iny}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-iny}\,dy\right) = \\ &= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-iny}\Big|_{-\pi}^{\pi}\right) = \\ &= \frac{L}{2\pi^2}\left(-\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{-in\pi} - \frac{1}{n^2}e^{in\pi}\right) = \\ &= \frac{(-1)^n L}{2\pi^2}\left(-\frac{\pi}{in} - \frac{\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right) \\ &= -\frac{(-1)^n L}{in\pi} = i\,\frac{(-1)^n L}{n\pi} \end{aligned}$$

and for $n = 0$ is

$$c_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = \frac{x^2}{4L}\Big|_{-L}^{L} = 0.$$

Therefore the complex Fourier series of f is

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\,\frac{(-1)^n L}{n\pi} e^{inx}.$$

Look at the next page!

The formula relating the real coefficients to the complex coefficients is

$$\begin{cases} a_0 = c_0 \\ a_n = c_n + c_{-n} \quad (n \geqslant 1) \\ b_n = i\,(c_n - c_{-n}) \end{cases}$$

and substituting we get indeed

$$\begin{cases} a_0 = c_0 = 0 \\ a_n = c_n + c_{-n} = i\,\frac{(-1)^n L}{n\pi} - i\,\frac{(-1)^n L}{n\pi} = 0 \\ b_n = i(c_n - c_{-n}) = i\left(i\,\frac{(-1)^n L}{n\pi} + i\,\frac{(-1)^n L}{n\pi}\right) = (-1)^{n+1}\frac{2L}{n\pi} \end{cases}$$

which is what we expected.

4. Find the solution $y : [0, \infty) \to \mathbb{R}$ of the following integral equation:

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t + e^t.$$

Solution:

Let us start by rewriting

$$\int_0^t y(\tau)\cosh(t-\tau)d\tau = \big(y * \cosh\big)(t).$$

Now we apply the Laplace transform on the ODE. The left-hand side is:

$$\mathcal{L}(y) + \mathcal{L}(y * \cosh(t)) = Y(s) + Y(s)\mathcal{L}(\cosh(t)) = Y(s) + \frac{s}{s^2-1}Y(s) = \frac{s^2+s-1}{s^2-1}Y(s)$$

The right-hand side is:

$$\mathcal{L}(t + e^t) = \frac{1}{s^2} + \frac{1}{s-1} = \frac{s^2+s-1}{s^2(s-1)}.$$

Setting these expressions equal and solving for $Y(s)$ gives:

$$Y(s) = \frac{s+1}{s^2} = \frac{1}{s} + \frac{1}{s^2},$$

and the solution is the obtained by applying the inverse Laplace transform:

$$y(t) = 1 + t.$$

Please turn!

5. Find the inverse Laplace transform of

$$\frac{s}{(s^2-16)^2},$$

by using

a) the differentiation rule: $\mathcal{L}'(f) = -\mathcal{L}(tf(t))$.

Solution:

To use the differentiation rule we need to recognise our function as a derivative of a Laplace transform. In fact:

$$\frac{s}{(s^2-16)^2} = -\frac{1}{2}\left(\frac{1}{s^2-16}\right)'$$

And this function is the Laplace transform of

$$-\frac{1}{2}\left(\frac{1}{s^2-16}\right) = -\frac{1}{8}\left(\frac{4}{s^2-16}\right) = -\frac{1}{8}\left(\mathcal{L}(\sinh(4t))\right).$$

Hence

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2-16)^2}\right) = -\frac{1}{8}\mathcal{L}^{-1}\left((\mathcal{L}(\sinh(4t))'\right) = \frac{1}{8}t\sinh(4t).$$

b) the integration rule: $\int_s^{+\infty}\mathcal{L}(f)(s')\,ds' = \mathcal{L}\left(\frac{f(t)}{t}\right)(s)$.

Solution:

We call $f(t)$ the inverse Laplace transform of this function. We plug the function into the integral on the left-hand side of the integral equation. This gives

$$\int_s^{+\infty}\mathcal{L}(f)(s')\,ds' = \int_s^{+\infty}\frac{s'}{((s')^2-16)^2}ds' = -\frac{1}{2}\left(\frac{1}{(s')^2-16}\right)\Big|_s^{+\infty} = \frac{1}{2}\left(\frac{1}{s^2-16}\right).$$

According to the integration rule this is equal to the Laplace transform of

$$\frac{1}{2}\left(\frac{1}{s^2-16}\right) = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

But we recognise this as the Laplace transform of:

$$\frac{1}{2}\left(\frac{1}{s^2-16}\right) = \frac{1}{8}\left(\frac{4}{s^2-16}\right) = \frac{1}{8}\mathcal{L}(\sinh(4t)).$$

Finally we get

$$\frac{f(t)}{t} = \frac{1}{8}\sinh(4t),$$

which agrees with the result we found before.