Prof. F. Da Lio
ETH Zürich

Analysis III

(stefano.dalesio@math.ethz.ch)

Solutions Serie 9

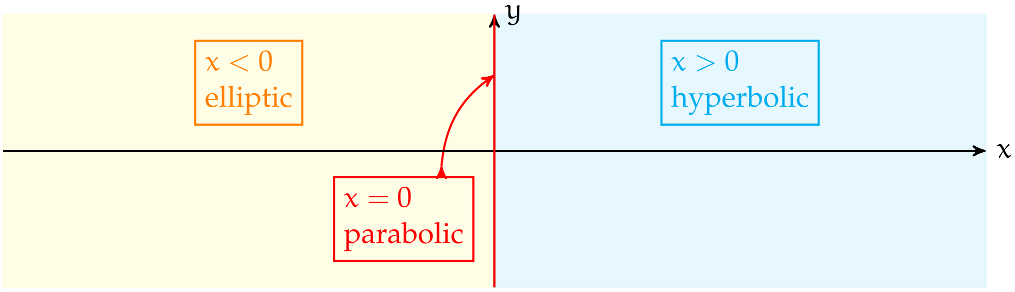
Theory reminder on the classification of PDEs: A 2nd order PDE is an equation of the form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

where the coefficients A, B, C may also be functions of x, y. We say that the PDE is, respectively, hyperbolic, parabolic or elliptic, if the function $AC - B^2$ is, respectively, always smaller, equal, or greater than zero. When the sign changes in different regions of the plane (x, y), the equation is called *of mixed type*.
For example the Euler-Tricomi equation

$$u_{xx} - xu_{yy} = 0$$

has $AC - B^2 = 1 \cdot (-x) - (0)^2 = -x$, and therefore is of mixed type: hyperbolic in the half plane $x > 0$, elliptic in the other half plane $x < 0$, and parabolic on the line $x = 0$.



Solutions Serie 9

Please turn!

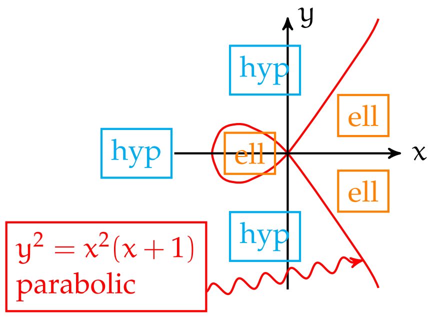
1. Consider the following PDEs - in what follows, $u = u(x, y)$ is a function of two variables.

$$u_{xx} + 2u_{xy} + u_{yy} + 3u_x + xu = 0, \tag{1}$$
$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0, \tag{2}$$
$$u_{xx} + 8u_{xy} + 2u_{yy} + e^x u_x = 0, \tag{3}$$
$$yu_{xx} + 2xu_{xy} + u_{yy} - u_y = 0, \tag{4}$$
$$(x+1)u_{xx} + 2yu_{xy} + x^2 u_{yy} = 0. \tag{5}$$

Which of this is hyperbolic? Parabolic? Elliptic? Of mixed type?
In the last case, try to understand in which region of the plane (x, y) they are hyperbolic, parabolic or elliptic[1].

Solution:

(1) $AC - B^2 = 1 - 1 = 0 \quad \rightsquigarrow \quad$ parabolic.
(2) $AC - B^2 = 2 - 1 = 1 > 0 \quad \rightsquigarrow \quad$ elliptic.
(3) $AC - B^2 = 2 - 4^2 = -14 < 0 \quad \rightsquigarrow \quad$ hyperbolic.
(4) $AC - B^2 = y - x^2 \quad \rightsquigarrow \quad$ of mixed type. The equation $y - x^2 = 0$ describes a parabola in the plane. Above the parabola the equation is elliptic, on the parabola it is parabolic, and below it is hyperbolic.
(5) $AC - B^2 = (x+1)x^2 - y^2 \quad \rightsquigarrow \quad$ of mixed type. The equation $(x+1)x^2 - y^2 = 0$ describes the following curve.



The closed region in the middle and the right region is where $(x+1)x^2 - y^2 > 0$ and therefore the PDE is elliptic. The big open region remaining is where the PDE is hyperbolic, while the red curve describes the points in which it is parabolic.

[1] You can plot the curve $\{AC - B^2 = 0\}$ on - say - Wolfram|Alpha to understand its shape.

2. Consider the following functions.

a) $u(x,t) = e^{-100t}\cos(2x)$

b) $u(x,t) = \sin(2x)\cos(8t)$

c) $u(x,t) = e^{-36t}\sin(3x)$

Which PDE between the heat equation, $u_t = c^2 u_{xx}$, and the wave equation, $u_{tt} = c^2 u_{xx}$, does each of these solve? Write down also which is the constant c in each case.

Solution:

a) $u(x,t) = e^{-100t}\cos(2x)$.

Solution:

We have

$$\begin{cases} u_t = -100e^{-100t}\cos(8x) = -100u \\ u_{tt} = \cdots = 100^2 u \\ u_{xx} = \cdots = -4u \end{cases}$$

To answer the question we are asked to compare either the first, or the second term, with a positive multiple of the third. This is possible (only) for the first term, and we need coefficient $c = 5$. Therefore u is a solution of the heat equation:

$$u_t = -100u = 5^2(-4u) = c^2 u_{xx}.$$

b) $u(x,t) = \sin(2x)\cos(8t)$.

Solution:

Here

$$\begin{cases} u_t = -8\sin(2x)\sin(8t) \\ u_{tt} = \partial_t(-8\sin(2x)\sin(8t)) = -64\sin(2x)\cos(8t) = -64u \\ u_{xx} = \cdots = -4u \end{cases}$$

In this case instead u is a solution of the wave equation, with coefficient $c = 4$:

$$u_{tt} = -64u = 4^2(-4u) = c^2 u_{xx}.$$

c) $u(x,t) = e^{-36t}\sin(3x)$.

Solution:

This is analogous to the first one.

$$\begin{cases} u_t = -36e^{-36t}\sin(3x) = -36u \\ u_{tt} = \cdots = 36^2 u \\ u_{xx} = \cdots = -9u \end{cases}$$

so we have a solution of the heat equation, this time with coefficient $c = 2$

$$u_t = -36u = 2^2(-9u) = c^2 u_{xx}.$$

3. Find the general solution $u = u(x, y)$ for the following PDEs:

a) $u_y + 2yu = 0$

Solution:

In this equation there are no derivatives in the variable x anywhere. We can thus think of this as a family of ODEs, one for each $x \in \mathbb{R}$. More formally for each fixed x the function $u(x, \cdot)$ must be a solution of

$$u_y + 2yu = 0.$$

We have already met equations of this kind and the general solution is given by some constant factor multiplied by $e^{-\int 2y} = e^{-y^2}$. Except that this time the 'constant' factor may depend on x, and is not constant anymore. The general solution is

$$u(x, y) = f(x)e^{-y^2},$$

where $f(x)$ is *any* function.

b) $u_{yy} = 4xu_y$.

Solution:

This is somehow similar because there are still no derivatives on x. First we call $v := u_y$ and we obtain the equation

$$v_y = 4xv.$$

As before this can be solved by solving the correspondent problem for $v(x, \cdot)$ for each fixed x. This time the general solution is

$$v(x, y) = f(x)e^{4xy},$$

and by integrating in y we get general solution

$$u(x, y) = g(x) + \frac{f(x)}{4x}e^{4xy}.$$

4. Consider the following time-dependent version of the heat equation on the interval $[0, L]$, in which the constant varies linearly with time. We also impose boundary conditions and we look for solutions:

$$u = u(x,t) \quad \text{s.t.} \quad \begin{cases} u_t = 2tc^2 u_{xx}, & x \in [0, L], t \in [0, +\infty) \\ u(0,t) = 0, & t \in [0, +\infty) \\ u(L,t) = 0, & t \in [0, +\infty) \end{cases}$$

Find all possible solutions of the specific form $u(x,t) = F(x)G(t)$.

Solution:

The differential equation becomes:

$$F(x)\dot{G}(t) = 2tc^2 F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{2tc^2 G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{2tc^2 G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(L,t) = F(L)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(L) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(L) = 0, \end{cases} \quad \text{and} \quad \dot{G}(t) = 2tkc^2 G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x} \right)$$

but then imposing the other condition:

$$0 = F(L) = C_1 \left(e^{\sqrt{k}L} - e^{-\sqrt{k}L} \right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}L} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}L \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(L) = C_1 L \quad \Leftrightarrow \quad C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

We impose the boundary conditions:

$$0 = F(0) = A \quad \Longrightarrow F(x) = B\sin(px)$$

and

$$0 = F(L) = B\sin(pL) \overset{(\text{if } B \neq 0)}{\Leftrightarrow} \quad pL = n\pi, \quad n \in \mathbb{Z}_{\geqslant 1}$$

Conclusion: we have a nontrivial solution for each $n \geqslant 1$, $k = k_n = -\frac{n^2\pi^2}{L^2}$:

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -2t\frac{n^2\pi^2c^2}{L^2}G$$

which has general solution

$$G_n(t) = C_n e^{-\frac{n^2\pi^2c^2}{L^2}t^2}.$$

The conclusion is that for every $n \geqslant 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = A_n e^{-\frac{n^2\pi^2c^2}{L^2}t^2} \sin\left(\frac{n\pi}{L}x\right), \qquad A_n \in \mathbb{R}.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose Fourier transform is

$$\sqrt{2\pi}\,\mathcal{F}(f)(\omega) = \frac{3}{(5+i\omega)}.$$

Compute the following integrals:

a) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$

Solution:

We can compute the integral of $f(x)$ using the value of the Fourier transform in $\omega = 0$:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \sqrt{2\pi}\,\mathcal{F}(f)(0) = \frac{3}{5}.$$

b) $\displaystyle\int_{-\infty}^{+\infty} xf(x)\,dx$, **c)** $\displaystyle\int_{-\infty}^{+\infty} x^2 f(x)\,dx$

Solution:

For any positive integer number $k \in \mathbb{Z}_{\geqslant 1}$, we would like to find the Fourier transform of $x^k f(x)$ and the compute it in $\omega = 0$. We can find this Fourier transform using the result of Exercise **5.b)** of Serie 8:

$$\mathcal{F}(xf)(\omega) = i\frac{d}{d\omega}\mathcal{F}(\omega) \quad \rightsquigarrow \quad \mathcal{F}(x^k f)(\omega) = i^k \frac{d^k}{d\omega^k}\mathcal{F}(f)(\omega).$$

So in general:

$$\int_{-\infty}^{+\infty} x^k f(x)\,dx = \sqrt{2\pi}\,\mathcal{F}(x^k f)(0) = \sqrt{2\pi}\,i^k \frac{d^k}{d\omega^k}\mathcal{F}(f)(0)$$

and all we need to do is compute the derivatives of $\frac{3}{(5+i\omega)}$:

- $$i\frac{d}{d\omega}\left(\frac{3}{(5+i\omega)}\right) = i\frac{-3i}{(5+i\omega)^2} = \frac{3}{(5+i\omega)^2}$$
- $$i^2\frac{d^2}{d\omega^2}\left(\frac{3}{(5+i\omega)}\right) = i\frac{d}{d\omega}\left(\frac{3}{(5+i\omega)^2}\right) = i\frac{-6i}{(5+i\omega)^3} = \frac{6}{(5+i\omega)^3}$$

Using what we said before we obtain:

- $$\int_{-\infty}^{+\infty} xf(x)\,dx = \left.\frac{3}{(5+i\omega)^2}\right|_{\omega=0} = \frac{3}{25}$$
- $$\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \left.\frac{6}{(5+i\omega)^3}\right|_{\omega=0} = \frac{6}{125}.$$

Remark: You can compute the inverse Fourier transform and find that the function $f(x)$ in the exercise is

$$f(x) = \begin{cases} 3e^{-5x}, & x \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

You can then verify that:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{0}^{+\infty} 3e^{-5x}\,dx = \frac{3}{5}$$

$$\int_{-\infty}^{+\infty} xf(x)\,dx = \int_{0}^{+\infty} 3xe^{-5x}\,dx = \frac{3}{25}$$

$$\int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_{0}^{+\infty} 3x^2 e^{-5x}\,dx = \frac{6}{125}$$

But also observe that computing these integrals in a traditional way is more difficult than the technique we used with the Fourier transform. In fact with the Fourier transform each x more in the integral amounts to computing one derivative more, while usually each x more in the integral amounts to make one more integration by parts.

We can quite easily compute the next derivatives and obtain the next integrals:

$$\int_{-\infty}^{+\infty} x^3 f(x)\,dx = \frac{18}{625}, \qquad \int_{-\infty}^{+\infty} x^4 f(x)\,dx = \frac{72}{3125}, \qquad \ldots$$

$$\boxed{\int_{-\infty}^{+\infty} x^k f(x)\,dx = 3 \cdot \frac{k!}{5^{k+1}}, \quad \forall k \in \mathbb{Z}_{\geqslant 0}}$$