D-MAVT
D-MATL

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Analysis III
Exam Solutions

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Exercise	1	2	3	4	5	Total
Value	8	10	8	8	14	48

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1. Laplace Transform (*8 Points*)

Find the solution $y : [0, +\infty) \to \mathbb{R}$ of the following integral equation:

$$y(t) + \frac{1}{\sqrt{2}} \int_0^t y(\tau) \sin(\sqrt{2}(t-\tau))\, d\tau = t \tag{1}$$

Solution:

We apply the Laplace transform to both sides of the integral equation (1) and obtain the following algebraic equation for the Laplace transform $Y = Y(s)$ of $y = y(t)$:

$$\begin{aligned}
&Y(s) + \frac{1}{\sqrt{2}} \mathcal{L}\left(y(t) * \sin(\sqrt{2}t)\right) = \frac{1}{s^2} \\
&Y(s) + \frac{1}{\cancel{\sqrt{2}}} Y(s) \cdot \frac{\cancel{\sqrt{2}}}{s^2+2} = \frac{1}{s^2} \\
&Y(s)\left(1 + \frac{1}{s^2+2}\right) = \frac{1}{s^2} \\
&Y(s) = \frac{s^2+2}{(s^2+3)s^2} = \frac{2}{3} \cdot \frac{1}{s^2} + \frac{1}{3} \cdot \frac{1}{(s^2+3)} \\
&\implies y(t) = \mathcal{L}^{-1}(Y(s))(t) = \frac{2}{3}t + \frac{\sin(\sqrt{3}t)}{3\sqrt{3}}
\end{aligned}$$

2. Fourier Series (*10 Points*)

a) (*5 Points*) Let $f(x) = x(\pi - x)$ for $x \in [0, \pi]$ and $f_{odd}(x)$ its odd, 2π-periodic extension. Compute the Fourier series of $f_{odd}(x)$.

Solution:

$f_{odd}(x)$ is an odd function, therefore its Fourier series contains only sines. The half period is $L = \pi$, so the Fourier series has the form

$$\sum_{n=1}^{+\infty} b_n \sin(nx)$$

with b_n given by

$$b_n = \frac{2}{\pi}\int_0^\pi f_{\text{odd}}(x)\sin(nx)dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)dx =$$
$$= \frac{2}{\pi}\cdot\left(\frac{(n^2x(x-\pi)-2)\cos(nx)+n(\pi-2x)\sin(nx)}{n^3}\right)\Bigg|_0^\pi = \frac{2}{\pi}\cdot\frac{2-2(-1)^n}{n^3} =$$
$$= \frac{4}{\pi n^3}(1-(-1)^n) = \begin{cases} 0, & n \text{ even} \\ \frac{8}{\pi n^3}, & n \text{ odd} \end{cases}$$

The Fourier series is

$$f_{\text{odd}}(x) \simeq \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(1-(-1)^n)}{n^3}\sin(nx) = \frac{8}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^3}\sin((2j+1)x)$$

b) (*5 Points*) Let α be a fixed number $0 < \alpha < \pi$, and $g_{\text{even}}(x)$ be the even, 2π-periodic extension of the function:

$$g(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \alpha \\ 0, & \alpha < x \leqslant \pi \end{cases}$$

Compute the Fourier series of $g_{\text{even}}(x)$.

Solution:

$g_{\text{even}}(x)$ is an even function, therefore its Fourier series contains only cosines. The half period is $L = \pi$, so the Fourier series has the form

$$a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx)$$

with a_0 and a_n given by

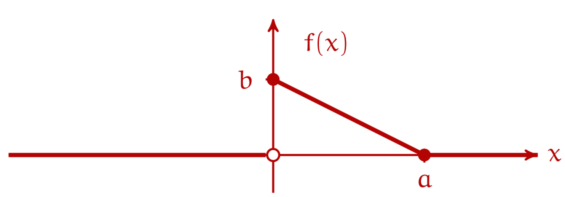
$$a_0 = \frac{1}{\pi}\int_0^\pi g_{\text{even}}(x)dx = \frac{1}{\pi}\int_0^\alpha dx = \frac{\alpha}{\pi}$$
$$a_n = \frac{2}{\pi}\int_0^\pi g_{\text{even}}(x)\cos(nx)dx = \frac{2}{\pi}\int_0^\alpha \cos(nx)dx = \frac{2}{\pi}\frac{\sin(n\alpha)}{n}$$

The Fourier series is

$$g_{\text{even}}(x) \simeq \frac{\alpha}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\sin(n\alpha)}{n}\cos(nx)$$

3. Fourier Integral (*8 Points*)

Let $a, b > 0$ be some fixed positive numbers and $f(x)$ be the following function:



See the next page!

a) (*1 Points*) Write a formula for $f(x)$.

Solution:

$$f(x) = \begin{cases} b\left(1-\frac{x}{a}\right), & 0 \leqslant x \leqslant a \\ 0, & \text{otherwise} \end{cases}$$

b) (*5 Points*) Compute its Fourier integral.

Solution:

The Fourier integral representation of $f(x)$ will be an expression of the form:

$$f(x) \simeq \int_0^{+\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega$$

where the coefficients $A(\omega), B(\omega)$ are

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(\omega v)dv = \frac{b}{\pi}\int_0^a \left(1-\frac{v}{a}\right)\cos(\omega v)dv =$$

$$= \frac{b}{\pi}\cdot\frac{\omega(a-v)\sin(\omega v)-\cos(\omega v)}{a\omega^2}\Bigg|_{v=0}^{v=a} = \frac{b}{\pi}\cdot\frac{-\cos(\omega a)+1}{a\omega^2} = \frac{b}{\pi a}\cdot\frac{1-\cos(\omega a)}{\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\sin(\omega v)dv = \frac{b}{\pi}\int_0^a \left(1-\frac{v}{a}\right)\sin(\omega v)dv =$$

$$= \frac{b}{\pi}\cdot\frac{-\omega(a-v)\cos(\omega v)-\sin(\omega v)}{a\omega^2}\Bigg|_{v=0}^{v=a} = \frac{b}{\pi}\cdot\frac{-\sin(\omega a)+\omega a}{a\omega^2} = \frac{b}{\pi a}\cdot\frac{a\omega-\sin(\omega a)}{\omega^2}$$

So the fourier integral is:

$$\frac{b}{\pi a}\int_0^{+\infty}\frac{(1-\cos(\omega a))\cos(\omega x)+(a\omega-\sin(\omega a))\sin(\omega x)}{\omega^2}\,d\omega$$

c) (*2 Points*) Find the value of the Fourier integral in the point $x = 0$. Motivate your answer.

Solution:

The Fourier integral coincides with the function wherever it is continuous, while in the points of discontinuity it is equal to the average of left and right limit of the function. The point $x = 0$ is a point of discontinuity, so the value of the Fourier integral at this point is

$$\frac{f(0^+)+f(0^-)}{2} = \frac{b+0}{2} = \frac{b}{2}$$

4. Wave Equation (*8 Points*)

Let $u = u(x,t)$ the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) = \begin{cases} -x^2+4\pi x-4\pi^2, & |x| \leqslant 2\pi \\ 0, & |x| > 2\pi \end{cases} \\ u_t(x,0) = g(x) = \begin{cases} \sin^2(x), & |x| \leqslant 2\pi \\ \frac{1}{x^2}, & |x| > 2\pi \end{cases} \end{cases}$$

Please turn!

a) (*4 Points*) Compute the value $u(0, \frac{\pi}{c})$.

Solution:

Using d'Alembert formula, the solution $u(x,t)$ of the wave equation is

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

In $(x,t) = (0, \frac{\pi}{c})$ the points $x \pm ct$ are $\pm\pi$. Therefore we need to use the first expression in the definition by cases of the functions f, g, and

$$u(0, \frac{\pi}{c}) = \frac{1}{2}(f(\pi) + f(-\pi)) + \frac{1}{2c}\int_{-\pi}^{\pi} g(s)ds = \frac{1}{2}(-\pi^2 - 9\pi^2) + \frac{1}{2c}\int_{-\pi}^{\pi} \sin^2(s)ds =$$
$$= -5\pi^2 + \frac{1}{2c}\pi$$

b) (*4 Points*) Compute the following asymptotic limit: $\lim_{a\to+\infty} u(a, \frac{a}{c})$.

Solution:

In the points of the form $(x,t) = (a, \frac{a}{c})$, the points $x \pm ct$ are, respectively, $x - ct = 0$ and $x + ct = 2a$, therefore

$$\lim_{a\to+\infty} u(a, \frac{a}{c}) = \lim_{a\to+\infty}\left(\frac{1}{2}(f(2a) + f(0)) + \frac{1}{2c}\int_0^{2a} g(s)ds\right) = -2\pi^2 + \frac{1}{2c}\int_0^{+\infty} g(s)ds =$$
$$= -2\pi^2 + \frac{1}{2c}\left(\int_0^{2\pi} \sin^2(s)ds + \int_{2\pi}^{+\infty} \frac{1}{s^2}ds\right) = -2\pi^2 + \frac{1}{2c}\left(\pi + \frac{1}{2\pi}\right).$$

5. Heat Equation (*14 Points*) Find the solution of the problem

$$\begin{cases} u_t = c^2 u_{xx}\,, & x \in [0,\pi], t > 0 \\ u(0,t) = u(\pi,t) = 0\,, & t > 0 \\ u(x,0) = \cos^2(x)\sin(x)\,, & x \in [0,\pi] \end{cases}$$

using the method of separation of variables, showing and motivating every step.

Solution:

With separation of variables $u(x,t) = F(x)G(t)$ the differential equation becomes:

$$F(x)\dot{G}(t) = c^2 F''(x)G(t),$$

which is better to write as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{c^2 G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality can be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{c^2 G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(\pi,t) = F(\pi)G(t) = 0 \quad \forall t \geqslant 0$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(\pi) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(\pi) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = c^2 kG(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \implies F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(\pi) = C_1 \left(e^{\sqrt{k}\pi} - e^{-\sqrt{k}\pi}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}\pi} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k}\pi \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \implies F(x) = C_1 x$$

and then

$$0 = F(\pi) = C_1 \pi \quad \implies C_1 = 0.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

$F(0) = 0$ if and only if $A = 0$. $F(\pi) = 0$ if and only if $B\sin(p\pi) = 0$, so if we want nontrivial solutions $B \neq 0$, we need to have

$$p = n$$

for some integer $n \geqslant 1$. In conclusion we have a nontrivial solution for each $n \geqslant 1$, $k = k_n = -n^2$:

$$F_n(x) = B_n \sin(nx)$$

The corresponding equation for $G(t)$ is

$$\dot{G} = -c^2 n^2 G$$

which has general solution

$$G_n(t) = C_n e^{-c^2 n^2 t}$$

The conclusion is that for every $n \geqslant 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin(nx)e^{-c^2 n^2 t}$$

and by the superposition principle:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(nx)e^{-c^2 n^2 t}$$

where the coefficients B_n are determined by the initial condition

$$g(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(nx).$$

This case is particularly easy because we can rewrite the datum at $t = 0$ as

$$\cos^2(x)\sin(x) = \frac{1}{2}(\cos(2x)+1)\sin(x) = \frac{1}{4}(\sin(3x) - \sin(x)) + \frac{1}{2}\sin(x) = \frac{1}{4}(\sin(3x) + \sin(x))$$

which is already expressed as a linear combination of these functions. There is no need to compute any Fourier series to obtain:

$$\begin{cases} B_1 = \frac{1}{4} \\ B_3 = \frac{1}{4} \\ \text{others } B_n = 0 \end{cases}$$

Finally, the solution will be

$$u(x,t) = \frac{1}{4}\sin(x)e^{-c^2 t} + \frac{1}{4}\sin(3x)e^{-9c^2 t}$$