D-MAVT
D-MATL

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Analysis III
Exam Solutions

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Exercise	1	2	3	4	5	Total
Value	8	10	10	14	8	50

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1. **Laplace Transform** (*8 Points*)
Solve, using the Laplace transform, the following initial value problem:

$$\begin{cases} y'' + 5y' + 6y = e^{-t}, & t \geqslant 0 \\ y(0) = 1, \\ y'(0) = -3. \end{cases} \tag{1}$$

Solution:

Let us denote by $Y = Y(s)$ the Laplace transform of $y(t)$. Then we use the formulas:

$$\begin{aligned} &\mathcal{L}(y'') = s^2Y - sy(0) - y'(0) = s^2Y - s + 3 \\ &\mathcal{L}(y') = sY - y(0) = sY - 1 \\ &\mathcal{L}(e^{-t}) = \frac{1}{s+1} \end{aligned}$$

and the initial value problem becomes the algebraic equation:

$$s^2Y - s + 3 + 5sY - 5 + 6Y = \frac{1}{s+1}$$

The left-hand side can be rewritten as

$$s^2Y - s + 3 + 5sY - 5 + 6Y = (s^2 + 5s + 6)Y - s - 2 = (s+2)(s+3)Y - (s+2)$$

Therefore the algebraic equation becomes

$$(s+2)(s+3)Y = s + 2 + \frac{1}{s+1} \quad \Leftrightarrow \quad Y = \frac{1}{s+3} + \frac{1}{(s+1)(s+2)(s+3)}$$

The first term is the Laplace transform of e^{-3t}.
For the second term we can either use partial fraction decomposition:

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2} \cdot \frac{1}{s+3} = \mathcal{L}\left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right)$$

Please turn!

or get the same result using the convolution product formula for exponentials

$$a \neq b \quad \Longrightarrow \quad e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a - b}$$

given in Solutions Serie 4 (Exercise **2.a)**) and:

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+2)(s+3)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) =$$
$$= e^{-t} * e^{-2t} * e^{-3t} = \left(e^{-t} - e^{-2t}\right) * e^{-3t} = \frac{e^{-t} - e^{-3t}}{2} - (e^{-2t} - e^{-3t}) =$$
$$= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$$

In any way we transform the second term, the solution will be

$$y(t) = \underbrace{e^{-3t}}_{\text{first term}} + \underbrace{\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}}_{\text{second term}} = \boxed{\frac{1}{2}e^{-t} - e^{-2t} + \frac{3}{2}e^{-3t}}$$

2. Fourier Series (*10 Points*)

Consider the function $f(x) = \left|\sin\left(\frac{x}{2}\right)\right|$.

a) (*2 Points*) Show that it is periodic of period 2π.

Solution:

For each $x \in \mathbb{R}$:

$$f(x+2\pi) = \left|\sin\left(\frac{x+2\pi}{2}\right)\right| = \left|\sin\left(\frac{x}{2}+\pi\right)\right| = \left|-\sin\left(\frac{x}{2}\right)\right| = \left|\sin\left(\frac{x}{2}\right)\right| = f(x)$$

$$\Longrightarrow \quad \underline{f \text{ is periodic of period } 2\pi}$$

b) (*6 Points*) Compute its Fourier series.

Solution:

$f(x)$ is a even function, therefore the coefficients $b_n = 0$.
For what follows it is important to remark that for $x \in [0, \pi]$ (actually for all $x \in [0, 2\pi]$) we have $|\sin(\frac{x}{2})| = \sin(\frac{x}{2})$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \overset{\text{(even)}}{=} \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right)\,dx = \frac{1}{\pi}\cdot 2 = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \overset{\text{(even)}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right)\cos(nx)\,dx =$$
$$= \frac{2}{\pi}\cdot\frac{2\left(2n\sin\left(\frac{x}{2}\right)\sin(nx) + \cos\left(\frac{x}{2}\right)\cos(nx)\right)}{4n^2-1}\Bigg|_0^{\pi} = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}$$

 See the next page!

The Fourier series is then

$$a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) = \boxed{\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} \cdot \cos(nx)}$$

c) (*2 Points*) Use the previous result to find the following numerical series:

$$\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} = ?$$

Solution:

First of all, the function $f(x)$ is continuous everywhere, therefore it coincides with its Fourier series:

$$\forall x \in \mathbb{R} : \qquad \left|\sin\left(\frac{x}{2}\right)\right| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} \cdot \cos(nx)$$

To calculate the above sum we want to get rid of $\cos(nx)$, which is easily done in the point $x = 0$, in which $\cos(nx) = 1$. The above equality then becomes:

$$\text{at } x = 0 : \qquad 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{1}{4n^2-1}$$

Which means that

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} = \frac{1}{2}}$$

3. Short Questionary (*10 Points*)

a) (*3 Points*) Consider the following 2^{nd} order PDEs and determine of which type (hyperbolic, parabolic, elliptic) they are:

(i) $u_{xx} + 6u_{xy} + 9u_{yy} = u + e^{-x}$

Solution:

$$AC - B^2 = 1 \cdot 9 - 3^2 = 9 - 9 = 0 \quad \Longrightarrow \quad \boxed{\text{parabolic}}$$

(ii) $2u_{xx} + 10u_{xy} + 15u_{yy} = u_x + u_y$

Solution:

$$AC - B^2 = 2 \cdot 15 - 5^2 = 30 - 25 = 5 > 0 \quad \Longrightarrow \quad \boxed{\text{elliptic}}$$

(iii) $u_{xx} + 2xu_{xy} - u_{yy} = \cosh(xy)$
Solution:

$$AC - B^2 = 1 \cdot (-1) - x^2 = -(1+x^2) < 0 \quad \Longrightarrow \quad \boxed{\text{hyperbolic}}$$

b) (*4 Points*) Let $u = u(x,t)$ be the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = 0, & x \in \mathbb{R} \\ u_t(x,0) = e^{-x^2}. & x \in \mathbb{R} \end{cases} \tag{2}$$

Find the following limit:

$$\lim_{a \to +\infty} u\left(a, \frac{a}{c}\right) = ?$$

Solution:

D'Alembert's formula for the wave equation tells us that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds = \frac{1}{2c} \int_{x-ct}^{x+ct} e^{-s^2}\, ds.$$

In particular

$$u\left(a, \frac{a}{c}\right) = \frac{1}{2c} \int_0^{2a} e^{-s^2}\, ds$$

and therefore

$$\lim_{a \to +\infty} u\left(a, \frac{a}{c}\right) = \frac{1}{2c} \int_0^{+\infty} e^{-s^2}\, ds = \frac{1}{2c} \cdot \frac{\sqrt{\pi}}{2} = \boxed{\frac{\sqrt{\pi}}{4c}}$$

c) (*3 Points*) Let $f(x)$ be a function with Fourier transform equal to:

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}.$$

Compute the integral:

$$\int_{-\infty}^{+\infty} f(x)\, dx = ?$$

Solution:

The Fourier transform is defined by

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x}\, dx$$

therefore in particular:

at $\omega = 0$: $\quad \widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, dx$

or equivalently we can compute the integral of the function as: $\quad \int_{-\infty}^{+\infty} f(x)\, dx = \sqrt{2\pi} \cdot \widehat{f}(0)$

In our case

$$\int_{-\infty}^{+\infty} f(x)\, dx = \sqrt{2\pi} \cdot \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}\bigg|_{\omega=0} = \boxed{2}$$

4. Heat Equation (*14 Points*)

Solve the following heat equation, using separation of variables and showing all the steps.

$$\begin{cases} u_t = c^2 u_{xx}, & x \in [0,1],\ t \geqslant 0 \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = g(x). & x \in [0,1] \end{cases} \tag{3}$$

where $g(x) = \sin(2\pi x) + 3\sin(5\pi x) + \sin(20\pi x)$.

Solution:

With separation of variables $u(x,t) = F(x)G(t)$ the differential equation becomes:

$$F(x)\dot{G}(t) = c^2 F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{c^2 G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality can be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{c^2 G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \geqslant 0$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \qquad \text{and} \quad \dot{G}(t) = c^2 k G(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x},$$

which is, however, not compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \Longrightarrow F(x) = C_1 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(1) = C_1 \left(e^{\sqrt{k}} - e^{-\sqrt{k}}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } e^{2\sqrt{k}} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \Longrightarrow F(x) = C_1 x$$

and then

$$0 = F(1) = C_1.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

$F(0) = 0$ if and only if $A = 0$. $F(1) = 0$ if and only if $B\sin(p) = 0$, so if we want nontrivial solutions $B \neq 0$, we need to have

$$p = n\pi$$

for some integer $n \geqslant 1$.

Conclusion: we have a nontrivial solution for each $n \geqslant 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x)$$

See the next page!

The corresponding equation for $G(t)$ is

$$\dot{G} = -c^2 n^2 \pi^2 G$$

which has general solution

$$G_n(t) = C_n e^{-c^2 n^2 \pi^2 t}$$

The conclusion is that for every $n \geqslant 1$ we have a solution

$$u_n(x,t) = F_n(x) G_n(t) = B_n \sin(n\pi x) e^{-c^2 n^2 \pi^2 t}$$

and by the superposition principle:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) e^{-c^2 n^2 \pi^2 t}$$

where the coefficients B_n are determined by the initial condition

$$g(x) = u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x).$$

This case is particularly easy because

$$g(x) = \sin(2\pi x) + 3\sin(5\pi x) + \sin(20\pi x)$$

is already expressed as a linear combination of these functions. There is no need to compute any Fourier series to obtain:

$$\begin{cases} B_2 = 1 \\ B_5 = 3 \\ B_{20} = 1 \\ \text{others } B_n = 0 \end{cases}$$

Finally, the solution will be

$$\boxed{u(x,t) = \sin(2\pi x) e^{-4c^2\pi^2 t} + 3\sin(5\pi x) e^{-25c^2\pi^2 t} + \sin(20\pi x) e^{-400c^2\pi^2 t}}$$

5. Laplace Equation *(8 Points)*
Consider the following Laplace equation on a centered disk of radius R:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(x,y) = \frac{\pi}{R(R^2+\pi^2)}(x^2 + 2xy + y^2). & \partial D_R \end{cases} \tag{4}$$

a) (*3 Points*) Find the value in the center of the disk:

$$u(0,0) = ?$$

Solution:

By Poisson's integral formula the value in the center is the average of the function on the boundary:

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(\vartheta)\, d\vartheta$$

The function on the boundary is

$$f(\vartheta) = \frac{\pi}{R(R^2+\pi^2)} (R^2 \cos^2(\vartheta) + 2R^2 \cos(\vartheta)\sin(\vartheta) + R^2 \sin^2(\vartheta)) = \frac{R^{\cancel{2}}\pi}{\cancel{R}(R^2+\pi^2)}(1+\sin(2\vartheta))$$

therefore

$$u(0,0) = \frac{1}{2\pi} \cdot \frac{R\pi}{(R^2+\pi^2)} \cdot \int_0^{2\pi} (1+\sin(2\vartheta))\, d\vartheta = \frac{1}{\cancel{2\pi}} \cdot \frac{R\pi}{(R^2+\pi^2)} \cdot \cancel{2\pi} = \boxed{\frac{R\pi}{R^2+\pi^2}}$$

b) (*3 Points*) Find the maximum on the whole disk:

$$\max_{(x,y)\in D_R} u(x,y) = ?$$

Solution:

By the maximum principle the maximum is reached on the boundary, so we need to

find the maximum, in the interval $\vartheta \in [0, 2\pi]$ of the function $\qquad f(\vartheta) = \frac{R\pi}{R^2+\pi^2} \cdot (1+\sin(2\vartheta))$

This can be done in several ways, and the easiest is to observe that $\sin(2\vartheta)$ assumes all possible values of the sine function, in particular its maximum is equal to 1. Therefore

$$\max_{\vartheta\in[0,2\pi]} \frac{R\pi}{R^2+\pi^2} \cdot (1 + \underbrace{\sin(2\vartheta)}_{\text{maximum } =1}) = \frac{R\pi}{R^2+\pi^2} \cdot (1+1) = \boxed{\frac{2R\pi}{R^2+\pi^2}}$$

c) (*2 Points*) Find the unique $R > 0$ for which this maximum is equal to 1.

Solution:

We need to impose:

$$\frac{2R\pi}{R^2+\pi^2} = 1 \quad \Leftrightarrow \quad R^2 - 2R\pi + \pi^2 = 0 \quad \Leftrightarrow \quad (R-\pi)^2 = 0 \quad \Leftrightarrow \quad \boxed{R = \pi}$$