ETH Zürich
Prof. Dr. F. Da Lio

Summer 2020
17 August

Exam Analysis III
D–MAVT, D–MATL

Exam Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	10		
3	8		
4	8		
5	14		
Total	48		

Please do not fill!

Completeness	

Before the exam:

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.

During the exam, please:

- Start every exercise on a new piece of paper.
- Put your **exam number** on the top right corner of every page.
- Motivate your answers. Write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with pencils. Please avoid using **red** or **green** ink pens.

After the exam:

- Make sure that every solutions sheet has your exam number on it.
- Place back the exam sheets, together with your solutions, in the envelope.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

Laplace Transforms: $\quad (F = \mathcal{L}(f))$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

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1. Laplace Transform (*8 Points*)

Find the solution $y : [0, +\infty) \to \mathbb{R}$ of the following integral equation:

$$y(t) + \frac{1}{\sqrt{2}} \int_0^t y(\tau) \sin(\sqrt{2}(t-\tau))\, d\tau = t \tag{1}$$

2. Fourier Series (*10 Points*)

a) (*5 Points*) Let $f(x) = x(\pi - x)$ for $x \in [0, \pi]$ and $f_{\text{odd}}(x)$ its odd, 2π-periodic extension. Compute the Fourier series of $f_{\text{odd}}(x)$.

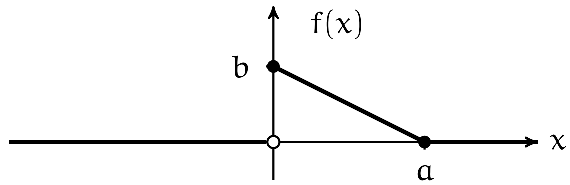
b) (*5 Points*) Let α be a fixed number $0 < \alpha < \pi$, and $g_{\text{even}}(x)$ be the even, 2π-periodic extension of the function:

$$g(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \alpha \\ 0, & \alpha < x \leqslant \pi \end{cases}$$

Compute the Fourier series of $g_{\text{even}}(x)$.

3. Fourier Integral (*8 Points*)

Let $a, b > 0$ be some fixed positive numbers and $f(x)$ be the following function:



a) (*1 Points*) Write a formula for $f(x)$.

b) (*5 Points*) Compute its Fourier integral.

c) (*2 Points*) Find the value of the Fourier integral in the point $x = 0$. Motivate your answer.

4. Wave Equation (*8 Points*)

Let $u = u(x,t)$ the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) = \begin{cases} -x^2 + 4\pi x - 4\pi^2\,, & |x| \leqslant 2\pi \\ 0\,, & |x| > 2\pi \end{cases} \\ u_t(x,0) = g(x) = \begin{cases} \sin^2(x)\,, & |x| \leqslant 2\pi \\ \frac{1}{x^2}\,, & |x| > 2\pi \end{cases} \end{cases}$$

a) (*4 Points*) Compute the value $u(0, \frac{\pi}{c})$.

b) (*4 Points*) Compute the following asymptotic limit: $\lim\limits_{a \to +\infty} u(a, \frac{a}{c})$.

5. Heat Equation (*14 Points*)

Find the solution of the problem

$$\begin{cases} u_t = c^2 u_{xx}\,, & x \in [0,\pi], t > 0 \\ u(0,t) = u(\pi,t) = 0\,, & t > 0 \\ u(x,0) = \cos^2(x)\sin(x)\,, & x \in [0,\pi] \end{cases}$$

using the method of separation of variables, showing and motivating every step.
[*Hint: It is convenient to rewrite the initial datum $u(x,0)$ as a linear combination of sines functions using elementary trigonometric identities.*]