ETH Zürich
Prof. Dr. F. Da Lio

Winter 2020
25 January

Exam Analysis III
D–MAVT, D–MATL

Please do not fill!

Exam Nr.:	

Please fill!

Surname:	
First Name:	
Student Card Nr.:	

Please do not fill!

Total points:	/ 50

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Exam Nr.:	

Please fill!

Last 6 digits of Student Card Nr.:	

Please do not fill!

Exercise	Value	Points	Control
1	8		
2	10		
3	10		
4	14		
5	8		
Total	50		

Please do not fill!

Completeness	

Please turn!

Important: Before the exam starts, please

- Turn off any mobile phone/device and place it inside your bag.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (**Legi**) on the desk.
- **Fill in** the front page of the exam **with your generalities (surname, first name, student card Nr.)**.
- **Fill in** the second page of the exam **with the last 6 digits of your student card Nr.**

During the exam, please

- Write your solutions on your sheets of paper. Do not write anything on the exam sheet!
- Start every exercise on a new piece of paper.
- Put your **exam Nr.** on the top right corner of every page.
- You are expected to motivate your answers. Please write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with **pencils**. **Do not** use any **red** or **green** ink pens.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German, or English-your mother tongue) dictionary.
- **No** further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

Laplace Transforms: $\big(F = \mathcal{L}(f)\big)$

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a),\ a \geqslant 0$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a),\ a \geqslant 0$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

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1. Laplace Transform (*8 Points*)
Solve, using the Laplace transform, the following initial value problem:

$$\begin{cases} y'' + 5y' + 6y = e^{-t}, \quad t \geqslant 0 \\ y(0) = 1, \\ y'(0) = -3. \end{cases} \tag{1}$$

2. Fourier Series (*10 Points*)

Consider the function $f(x) = \left|\sin\left(\frac{x}{2}\right)\right|$.

a) (*2 Points*) Show that it is periodic of period 2π.

b) (*6 Points*) Compute its Fourier series.

c) (*2 Points*) Use the previous result to find the following numerical series:

$$\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} = ?$$

Please turn!

3. Short Questionary (*10 Points*)

a) (*3 Points*) Consider the following 2^{nd} order PDEs and determine of which type (hyperbolic, parabolic, elliptic) they are:

(i) $u_{xx} + 6u_{xy} + 9u_{yy} = u + e^{-x}$

(ii) $2u_{xx} + 10u_{xy} + 15u_{yy} = u_x + u_y$

(iii) $u_{xx} + 2xu_{xy} - u_{yy} = \cosh(xy)$

b) (*4 Points*) Let $u = u(x,t)$ be the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = 0, & x \in \mathbb{R} \\ u_t(x,0) = e^{-x^2}. & x \in \mathbb{R} \end{cases} \tag{2}$$

Find the following limit:

$$\lim_{a \to +\infty} u\left(a, \frac{a}{c}\right) = ?$$

c) (*3 Points*) Let $f(x)$ be a function with Fourier transform equal to:

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}.$$

Compute the integral:

$$\int_{-\infty}^{+\infty} f(x)\, dx = ?$$

4. Heat Equation (*14 Points*)

Solve the following heat equation, using separation of variables and showing all the steps.

$$\begin{cases} u_t = c^2 u_{xx}, & x \in [0,1],\ t \geqslant 0 \\ u(0,t) = u(1,t) = 0, & t \geqslant 0 \\ u(x,0) = g(x). & x \in [0,1] \end{cases} \tag{3}$$

where $g(x) = \sin(2\pi x) + 3\sin(5\pi x) + \sin(20\pi x)$.

5. Laplace Equation (*8 Points*)
Consider the following Laplace equation on a centered disk of radius R:

$$\begin{cases} \nabla^2 u = 0, & D_R \\ u(x,y) = \frac{\pi}{R(R^2+\pi^2)}(x^2 + 2xy + y^2). & \partial D_R \end{cases} \tag{4}$$

a) (*3 Points*) Find the value in the center of the disk:

$$u(0,0) = ?$$

b) (*3 Points*) Find the maximum on the whole disk:

$$\max_{(x,y)\in D_R} u(x,y) = ?$$

c) (*2 Points*) Find the unique $R > 0$ for which this maximum is equal to 1.