# Complex Analysis Formulary

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Autumn semester 2019

Throughout the exam, let $\mathsf{D}_{z_0}(r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\mathsf{D}^*_{z_0}(r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$, $z_0 \in \mathbb{C}$, $r > 0$. By a domain $D \subset \mathbb{C}$, we mean an open and connected subset.

**Theorem (Cauchy-Riemann equation)** Let $U \subset \mathbb{C}$ be open and $u, v : U \to \mathbb{R}$ be differentiable functions such that the partial derivatives $\partial_x u, \partial_y u, \partial_x v$ and $\partial_y v$ are continuous at $z_0 \in U$. Then $f = u + iv$ is complex differentiable at $z_0$ if and only if

$$\partial_x u = \partial_y v \text{ and } \partial_y u = -\partial_x v, \text{ at the point } z_0.$$

**Theorem** Let $D$ be a domain. For a continuous function $f : D \to \mathbb{C}$, the following statements are equivalent:

(a) There exist a primitive $F$ for $f$.

(b) For every piecewise $C^1$ path $\gamma$ in $D$, the integral $\int_\gamma f(z)dz$ only depends on the end points of $\gamma$.

(c) $\int_\gamma f(z)dz = 0$ for every closed piecewise $C^1$ path $\gamma$ in $D$.

**Theorem (Cauchy)** Let $f : D \to \mathbb{C}$ be holomorphic in the domain $D$. Let $\gamma \subset D$ be a closed $C^1$-path which is homotopic in $D$ to a constant path. Then

$$\int_\gamma f(z)dz = 0.$$

**Theorem (Morera)** Let $f : D \to \mathbb{C}$ be a continuous function in the domain $D$. Assume that for every (solid) rectangle $R \subset \mathsf{D}$ whose sides are parallel to the $x - y$ axes we have $\int_{\partial R} f(z)dz = 0$. Then $f$ is holomorphic in $D$.

**Theorem (Cauchy integral formula)** Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a holomorphic function. Let $\overline{\mathsf{D}} \subset U$ be a disk. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathsf{D}} \frac{f(w)}{w - z} dw, \quad \forall z \in \mathsf{D}.$$

For the $k$-th derivative $f^{(k)}$ of $f$, we have

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial \mathsf{D}} \frac{f(w)}{(w - z)^{k+1}} dw.$$

**Theorem (Estimate of derivatives)** Suppose that a function $f$ is holomorphic in an open disk $\mathsf{D}_{z_0}(r)$ and that $|f(z)| \leq N$ on $\mathsf{D}_{z_0}(r)$. Then for each $k \geq 0$, $z \in \mathsf{D}_{z_0}(r)$:

$$|f^{(k)}(z)| \leq \frac{k! N r}{(r - |z - z_0|)^{k+1}}.$$

**Theorem (Liouville)** The only bounded entire functions are the constant functions on $\mathbb{C}$.

**Theorem (Schwarz's lemma)** Suppose that a function $f$ is holomorphic in the unit disk $\mathsf{D}_0(1)$ and $f(0) = 0, |f(z)| \leq 1$ for every $z \in \mathsf{D}_0(1)$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for every $z \in \mathsf{D}_0(1)$. Moreover, if $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $0 \neq z_0 \in \mathsf{D}_0(1)$, then $f(z) = cz$ for some $|c| = 1$.

**Theorem (Maximum principle)** 1) Let $f : D \to \mathbb{C}$ be a holomorphic function in the domain $D$. If $|f|$ has a local maximum at an (interior) point of $D$, then $f$ is constant
2) Let $D$ be a bounded domain, $f : \overline{D} \to \mathbb{C}$ continuous and holomorphic in $D$. Then $|f|$ obtains its maximal value at some point on the boundary $\partial D$ of $D$.

**Theorem (Riemann Extension)** Let $f : \mathsf{D}^*_{z_0}(r) \to \mathbb{C}$ be a holomorphic function. Then $z_0$ is a removable singularity of $f$ if and only if $f$ is bounded in some punctured disk centered at $z_0$.

**Theorem(Existence of a branch)** Suppose that a function $f$ is holomorphic and free of zeros in a domain $D$. There exists a branch of $log\, f(z)$ in $D$ if and only if

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed, piecewise $C^1$ path $\gamma \subset D$.

**Theorem (Limit of holomorphic functions)** Suppose that a sequence of holomorphic functions $\{f_n\}$ on a domain $D$ converges locally uniformly to the function $f$ on $D$. Then $f$ is a holomorphic function on $D$. Moreover, $f_n^{(k)} \to f^{(k)}$ locally uniformly on $D$ for each positive integer $k$.

**Theorem (Taylor series)** Suppose that $f$ is a holomorphic function on a domain $U$, that $z_0 \in U$ and that the open disk $\mathsf{D}_{z_0}(r)$ is contained in $D$. Then $f$ can be represented in $\mathsf{D}_{z_0}(r)$ as the sum of a Taylor series centered at $z_0$. This expansion is uniquely determined by $f$: if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in $\mathsf{D}_{z_0}(r)$, the coefficient $a_n$ is given by $a_n = f^{(n)}(z_0)/n!$.

**Theorem (Casorati-Weierstrass)** If a holomorphic function $f : \mathsf{D}^*_{z_0}(r) \to \mathbb{C}$ has an essential singularity at $z_0$, then its image is dense in $\mathbb{C}$.

**Theorem (Residue theorem)** Let $D \subset \mathbb{C}$ be a domain and $z_1, \cdots, z_n \in D$ be a finite collection of distinct points. Let $f : D \setminus \{z_1, \cdots, z_n\} \to \mathbb{C}$ be a holomorphic function. Let $\gamma \subset D \setminus \{z_1, \cdots, z_n\}$ be a closed piecewise $C^1$-path. Then

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} I_\gamma(z_k) Res_{z_k}(f).$$

**Theorem (Argument principle)** Let $\gamma = \partial D$ be a Jordan path bounding a bounded domain $D$, oriented counter clockwise. Let $f$ be a meromorphic function in a domain $U$ that contains $\overline{D}$. Assume $f$ has no zeros and no poles on $\gamma$. Denote by $z_1, \cdots, z_r \in D$ the zeros of $f$ in $D$ and by $n_k$ the order of the zero $z_k$, $k = 1, \cdots, r$. Denote by $w_1, \cdots, w_s$ the poles of $f$ in $D$ and by $m_j$ the order of the pole $w_j$,

$j = 1, \cdots, s$. Then

$$I_{f \circ \gamma}(0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{r} n_k - \sum_{j=1}^{s} m_j.$$

**Theorem (Rouché)** Let $\gamma = \partial D$ be a Jordan path bounding a bounded domain $D$. Let $U \subset \mathbb{C}$ be domain containing $\overline{D}$ and $f, g : U \to \mathbb{C}$ be holomorphic functions. Assume that for all $z \in \gamma$,

$$|f(z) - g(z)| < |f(z)| + |g(z)|.$$

Then $f$ and $g$ have the same number of zeros in $D$ when counted with multiplicities.

**Theorem (Open mapping)** If $f : D \to \mathbb{C}$ is a nonconstant holomorphic function in the domain $D$, then for any open subset $U \subset D$, $f(U)$ is open in $\mathbb{C}$.